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The arbelos in Wasan geometry, Shinohara's problem

Hiroshi Okumura
Takahanadai Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. A configuration arising from a problem of the arbelos in Wasan geometry proposed by Shinohara is considered.

Keywords. arbelos, Wasan geometry, Shinohara's problem.

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1. Introduction

In this note we consider a configuration arising from a problem of the arbelos in Wasan geometry. A similar configuration can be found in [1]. Let us consider an arbelos formed by the three semicircles α, β and γ with diameters BC, CA and AB, respectively for a point C on the segment AB (see Figure 1). The radii of α, β and γ are denoted by a, b and c, respectively. Let ε_1, ε_2, $\cdots$, ε_n be congruent circles of radius $r < a$ touching the radical axis of α and β from the side opposite to A such that ε_1 touches α externally, ε_k $(k = 2, 3, \cdots, n)$ touches ε_{k-1} at the farthest point on ε_{k-1} from AB, and ε_n touches γ internally. We denote the arbelos with the circles ε_1, ε_2, $\cdots$, ε_n by $\mathcal{S}(n)$. Shinohara (篠原善成) considered the following problem in 1812 [2] (see Figure 2).

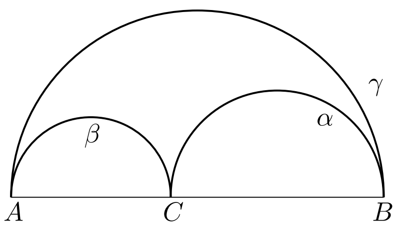

Figure 1.

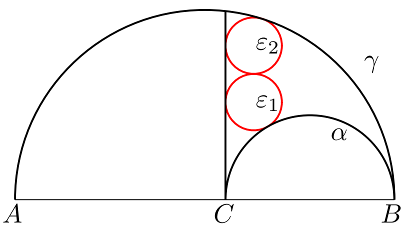

Figure 2: $\mathcal{S}(n)$ (n=2)

Problem 1. Find c in terms of a and r for the configuration $\mathcal{S}(2)$.

2. Solution of the problem

Shinohara gave the following solution of Problem 1[2].

$$c = \frac{a^2 + 2r\sqrt{ar} + r^2}{a - r}. \tag{1}$$

He also gave the values of a, c and r such that $(a, c, r) = (4, 7, 1)$ satisfying (1). Let F be the foot of perpendicular from the center of ε_1 to AB. The next proposition gives a solution of Problem 1, since $c = a + b$.

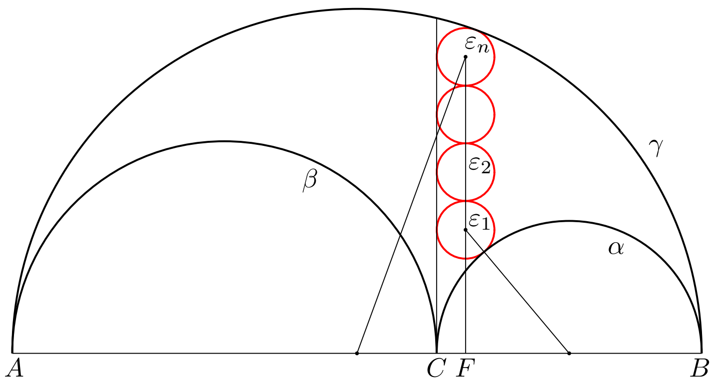

Figure 3.

Proposition 1. *The following relation holds for $\mathcal{S}(n)$.*

$$b = \frac{r\left(\sqrt{a} + (n-1)\sqrt{r}\right)^2}{a - r}. \tag{2}$$

Proof. The distance between the center of ε_1 and F equals $2\sqrt{ar}$ by the right triangle made by the centers of α and ε_1 and the point F (see Figure 3). The distance from the center of ε_n to F equals $\sqrt{(a+b-r)^2 - (r-(a-b))^2} = 2\sqrt{(a-r)b}$ by the right triangle made by the centers of γ and ε_n and F. Therefore we get $2\sqrt{ar} + 2(n-1)r = 2\sqrt{(a-r)b}$. Solving the last equation for b, we have (2). □

Let E be the farthest point on ε_n from F for $\mathcal{S}(n)$ (see Figure 4). We have

$$|EF| = 2\sqrt{ar} + (2n-1)r. \tag{3}$$

Hence $|EF| = 4 + 3 = 7 = c$ if $(a, c, r) = (4, 7, 1)$ for $\mathcal{S}(2)$. Therefore F coincides with the center of γ in the case given by Shinohara (see Figure 5).

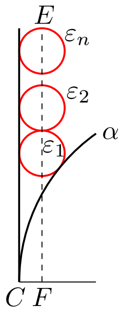

Figure 4.

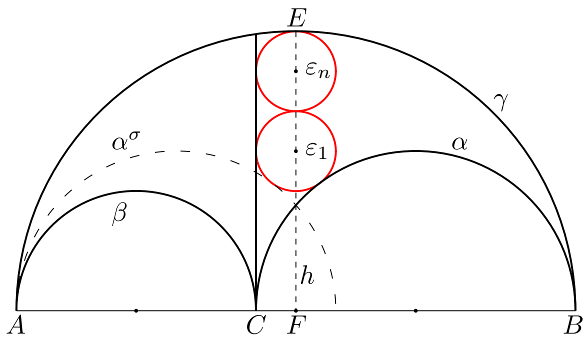

Figure 5: $\mathcal{T}(2)$, $(a, c, r) = (4, 7, 1)$.

[2] There are writing errors in the text describing c in terms of a and r in [2].

3. A configuration arising from the problem

Let h be the perpendicular to AB at the center of γ. Let σ be the reflection in the line h. If the semicircle α^σ touches ε_1 externally for $\mathcal{S}(n)$, then the configuration is denoted by $\mathcal{T}(n)$. Hence $\mathcal{S}(2)$ in the Shinohara's case is $\mathcal{T}(2)$. We will show that the configuration $\mathcal{T}(n)$ has interesting properties, which deserves attention. We use the next proposition.

Proposition 2. *The following relation holds for $\mathcal{T}(n)$.*

$$a = \frac{1}{2}\left(2n+1+\sqrt{4n+1}\right)r. \tag{4}$$

Proof. Since the radius of γ equals $|EF|$ and $2a-r$, we have $2\sqrt{ar}+(2n-1)r = 2a-r$ by (3). Solving the equation for a, we get

$$a = \frac{1}{2}\left(2n+1\pm\sqrt{4n+1}\right)r.$$

However $a = \left(2n+1-\sqrt{4n+1}\right)r/2$ implies $c = 2a-r = \left(2n-\sqrt{4n+1}\right)r < 0$, a contradiction. Therefore we get (4). □

4. Integer case

In this section we consider the case in which the ratio a/r is an integer for $\mathcal{T}(n)$. The semicircle of diameter CC^σ erected on the same side of AB as ε_1 is denoted by ζ_0 (see Figure 6).

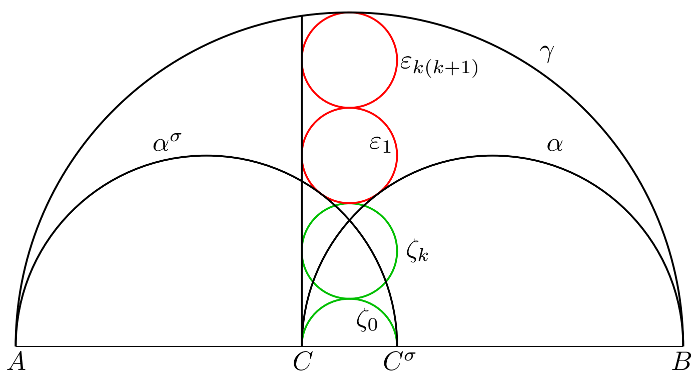

Figure 6: $\mathcal{T}(k(k+1))$ $(k=1)$.

Theorem 1. *The following statements hold for the configuration $\mathcal{T}(n)$.*
(i) *The ratio a/r is an integer if and only if $n = k(k+1)$ for a positive integer k.*
(ii) *In the event of* (i)*, we have $a/r = (k+1)^2$, and there are congruent circles ζ_1, $\zeta_2, \cdots, \zeta_k$ of radius r such that ζ_i $(i = 1,2,3,\cdots,k)$ touches ζ_{i-1} at the farthest point on ζ_{i-1} from AB and ζ_k touches ε_1 at the closest point on ε_1 to AB.*

Proof. If a/r is an integer, $4n+1$ is a square of an odd integer by (4). Therefore there is a positive integer k such that $4n+1 = (2k+1)^2$. The last equation implies $n = k(k+1)$. Conversely $n = k(k+1)$ implies $a/r = (k+1)^2$, which is also an integer. This proves (i). Since the radius of γ equals $2a-r$, the distance between the farthest point on the semicircle ζ_0 from AB and the closest point on ε_1 to AB equals

$$(2a-r)-(2nr+r) = (2(k+1)^2r-r)-(2k(k+1)r+r) = 2kr.$$

This proves (ii). □

Let t be the external common tangent of α and α^σ. We consider the case in which the internal common tangent of the circles ε_m and ε_{m+1}, which is parallel to AB, coincides with t for some integer m for $\mathcal{T}(n)$. This is equivalent to that ε_m touches t from the same side as the point C (see Figure 7).

Theorem 2. *For the configuration $\mathcal{T}(n)$, ε_m touches t from the same side as C for a positive integer m if and only if $n = 2p(2p+1)$ and $m = 2p^2$ for a positive integer p.*

Proof. Since γ has radius $2a - r$, the circle ε_m touches t from the same side as C if and only if $2a - r - 2(n-m)r = a$. By (4), this is equivalent to

$$m = \frac{1}{4}\left(2n + 1 - \sqrt{4n+1}\right). \tag{5}$$

We now assume that ε_m touches t from the same side as C. Then $4n+1$ is a square of an odd integer by (5). Hence there is a positive integer k such that $n = k(k+1)$ as just shown in the proof of Theorem 1. Then $m = k^2/2$. Hence k must be an even number, i.e., $k = 2p$ for a positive integer p. Therefore we have $n = 2p(2p+1)$ and $m = 2p^2$. Conversely if $n = 2p(2p+1)$ and $m = 2p^2$, then (5) holds. □

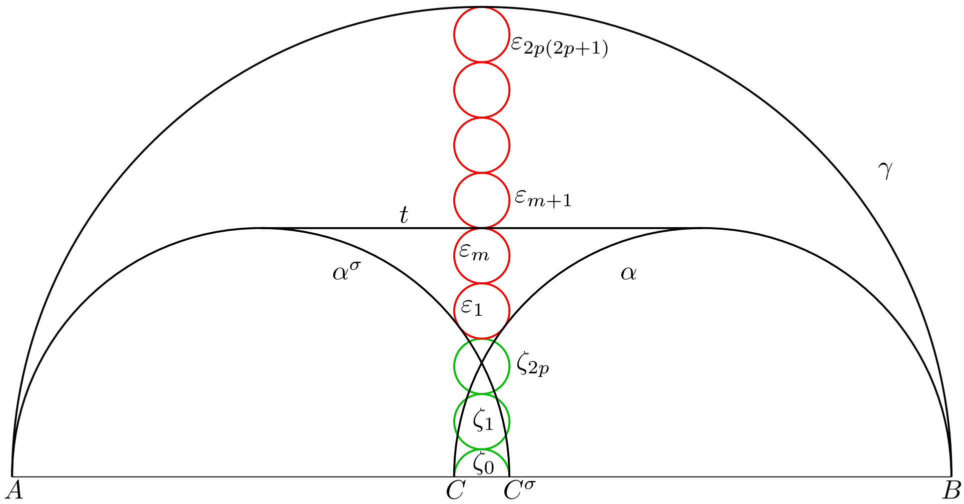

Figure 7: $\mathcal{T}(2p(2p+1))$, $m = 2p^2$ $(p = 1)$.

Figure 7 shows the case $p = 1$. The line t divides the set of the circles $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ into the two sets $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{2p^2}\}$ and $\{\varepsilon_{2p^2+1}, \varepsilon_{2p^2+2}, \cdots, \varepsilon_n\}$ in the event of Theorem 2. Since the number of the circles in the latter set equals $n - 2p^2 = 2p(p+1)$, the ratio of the numbers of the circles in the two sets equals $p : p+1$.

References

[1] H. Okumura, The arbelos in Wasan geometry, Nishimura's problem, Sangaku J. Math., **9** (2025) 1–4.

[2] Shimura (志村昌義) et al. ed., Kiōshū (淇澳集) volume 8, 1812 Tohoku University Digital Archives.