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The arbelos in Wasan geometry, Nishimura's problem

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Abstract. We consider an arbelos problems in Wasan geometry proposed by Nishimura.

Keywords. arbelos, Wasan geometry, Nishimura's problem

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1. Introduction

There are many problems involving an arbelos in Wasan geometry (Japanese old geometry developed in the Edo period). However most such problems are considering a special case in which the smaller two semicircles forming the arbelos are congruent (see Figure 1). Thereby we can generalize such problems by considering a general arbelos with no congruent semicircles as in [3, 4] (see Figure 2). In this note we also consider an arbelos problem with two congruent semicircles, however the general arbelos is not considered.

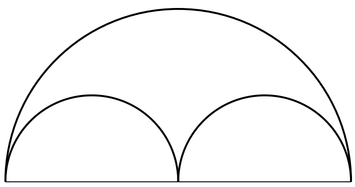

Figure 1.

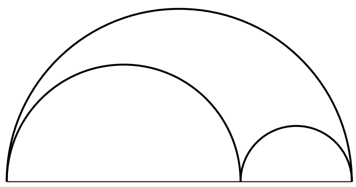

Figure 2.

2. The problem

We now consider an special arbelos formed by the three semicircles α, β and γ with diameters AM, BM and AB, respectively erected on the same side of the segment AB for the midpoint M of AB. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be congruent circles for a positive integer n such that ε_1 touches α and β externally, ε_i $(i = 2, 3, \cdots, n)$ touches ε_{i-1} at the farthest point on ε_{i-1} from M, and ε_n touches γ internally. The arbelos with the circles $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ is denoted by $\mathcal{N}(n)$ (see Figure 3). Let a and r be the radii of α and ε_1, respectively. We consider the following problem.

Problem 1. Find a in terms of r for the configuration $\mathcal{N}(n)$.

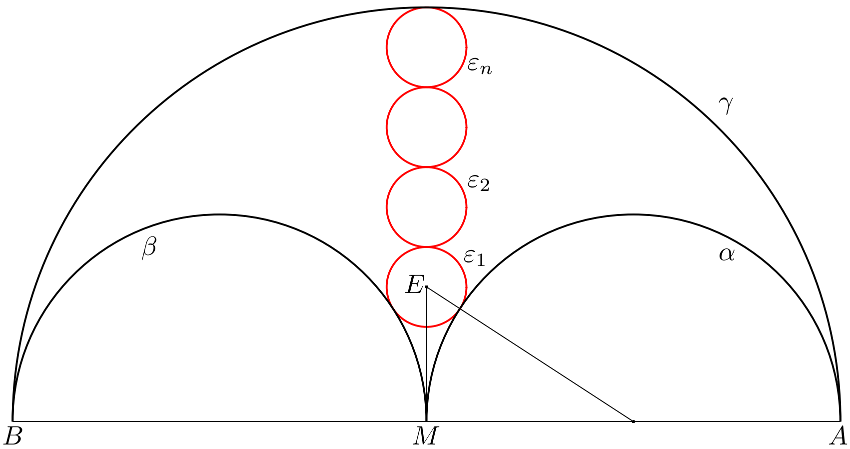

Figure 3: $\mathcal{N}(n)$, $n = 4$

Nishimura (西村永貞) proposed the problem in the case $n = 2, 3$ in 1831 [2]. Toyoyoshi (豊由周齋)[2] considered the case $n = 2$ [5]. A solution showing $a = 4r$ in the case $n = 3$ by Itō (伊東隷尾)[3] can be found in [1].

We give a solution of Problem 1. Let E be the center of the circle ε_1 (see Figure 3). From the right triangle formed by the points E and M and the center of α, we get $|EM| = \sqrt{(a+r)^2 - a^2}$, while $2a = |EM| + (2n-1)r$. Eliminating $|EM|$ from the two equations and solving the resulting equation for a, we get $a = (4n - 1 \pm \sqrt{8n+1})r/4$. But $a = (4n - 1 - \sqrt{8n+1})r/4$ implies $2a - 2nr = -(1 + \sqrt{8n+1})r/2 < 0$. Therefore we get

$$a = \frac{1}{4}\left(4n - 1 + \sqrt{8n+1}\right) r. \tag{1}$$

This is a solution of the problem.

3. Integer case

Let c be the radius of the semicircle γ. In this section we consider the case in which the ratio c/r is an integer.

Theorem 1. *For the configuration $\mathcal{N}(n)$, the ratio c/r is an integer if and only if $n = k(k+1)/2$ for a positive integer k. In this event $c/r = k(k+2)$ holds.*

[2] died in 1887.
[3] born in 1812.

Proof. From (1) we have

$$\frac{c}{r} = \frac{1}{2}\left(4n - 1 + \sqrt{8n+1}\right). \tag{2}$$

Hence if c/r is an integer, then $8n+1$ is a square of an odd integer, i.e., there is a non-negative integer k such that $8n+1 = (2k+1)^2$. The last equation implies $n = k(k+1)/2$. However this implies $n = 0$ if $k = 0$. Therefore k must be positive. Conversely if $n = k(k+1)/2$ for a positive integer k, then $c/r = k(k+2)$ by (2). This is also an integer. □

4. External common tangent of α and β

Let t be the external common tangent of the semicircles α and β. In the case c/r being an integer, we consider the case in which the internal common tangent of the circles ε_m and ε_{m+1}, which is parallel to AB, coincides with the line t for some integer m for the configuration $\mathcal{N}(n)$. This is equivalent to that ε_m touches t from the same side as the point M. We have the next theorem (see Figure 4).

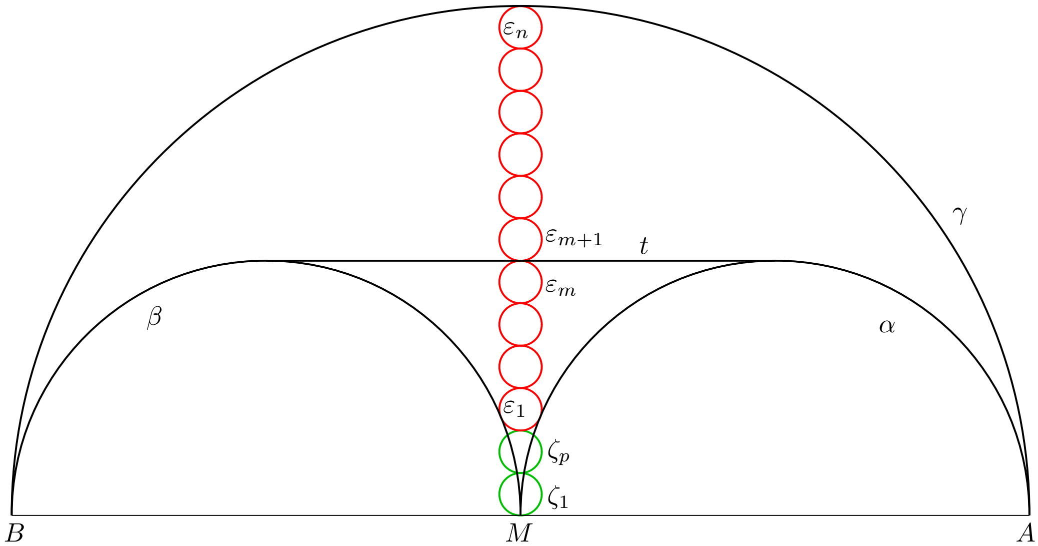

Figure 4: $\mathcal{N}(n)$, $n = p(2p+1)$, $m = p^2$ $(p = 2)$.

Theorem 2. *If c/r is an integer, then the following statements hold for $\mathcal{N}(n)$.*
(i) *The circle ε_m touches t from the same side as M if and only if $n = p(2p+1)$ and $m = p^2$ for a positive integer p.*
(ii) *In the event of* (i)*, there are circles $\zeta_1, \zeta_2, \cdots, \zeta_p$ congruent to ε_1 such that ζ_1 touches AB at M from the same side as ε_1, ζ_i $(i = 2, 3, \cdots, p)$ touches ζ_{i-1} at the farthest point on ζ_{i-1} from M and ζ_p touches ε_1 at the closest point on ε_1 to M.*

Proof. The circle ε_m touches t from the same side as M if and only if $2(n-m)r = a$. This is equivalent to $2mr = 2nr - a = (4n + 1 - \sqrt{8n+1})r/4$ by (1), i.e., it is equivalent to

$$m = \frac{4n + 1 - \sqrt{8n+1}}{8}. \tag{3}$$

Hence if ε_m touches t from the same side as M, then $\sqrt{8n+1}$ is an integer, and $n = k(k+1)/2$ for a positive integer k as in the proof of Theorem 1. Substituting the last equation in (3) and rearranging, we have $m = k^2/4$. Therefore k is an even number. Let $k = 2p$ for a positive integer p. Then $n = p(2p+1)$ and $m = p^2$. Conversely if $n = p(2p + 1)$ and $m = p^2$ for a positive integer p, then (3) holds. Therefore ε_m touches t from the same side as M. This proves (i). We prove (ii). Recall $c/r = k(k + 2)$ by Theorem 1. The distance between M and the closest point on ε_1 to AB equals

$$c - 2nr = k(k + 2)r - k(k + 1)r = kr = 2pr.$$

This proves (ii). □

Figure 4 shows the case $p = 2$. The line t divides the set of the circles $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ into two sets $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{p^2}\}$ and $\{\varepsilon_{p^2+1}, \varepsilon_{p^2+2}, \cdots, \varepsilon_n\}$ in the event of Theorem 2. Since the number of the circles in the latter set equals $n - p^2 = p(p + 1)$, the ratio of the numbers of the circles in the two sets equals $p : p + 1$.

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