Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 8 (2024), pp. 21–32
Received 1 October 2024. Published on-line 20 December 2024.
web: http://www.sangaku-journal.com/

A Summary of *Sanpo tai kajo no daishu* by Aida Yasuaki

J. MARSHALL UNGER
Emeritus Professor, The Ohio State University
3144 Gracefield Rd., Apt. T-09, Silver Spring, MD (USA)
e-mail: unger.26@osu.edu

Abstract. We describe an Edo period text intended to instruct the reader on how to reduce the number of multiplications needed to find the numerical solution of an equation, highlighting features of it that help us understand how mathematical knowledge was shared in early 19th century Japan and the degree of its sophistication in geometry.

Keywords. *wasan*, circle, sphere

Mathematics Subject Classification (2020). 01A27, 51M04

1. INTRODUCTION

The Yamagata University Library ascribes [1] to Aida Yasuaki (1747–1817), saying that this manuscript was corrected (*tei* 訂, presumably during copying) by Aida's disciple Watanabe Jiemon Kazu 渡辺治右衛門一 (1767–1819).[2] The theme of [1], expressed in this work's title—

[2] On the first page of the text, after the title, we find the lines *Saijō-ryū Aida Yasuaki henshū* 最上流 会田安明編集 and *Monsei Watanabe Jiemon Kazu* 門生 渡辺治衛門一. The Saijō (or Mogami)-ryū was the name of Aida's school of *wasan*. Many other collections of problems ascribed to Aida have only *hen* 'edited by' 編 (much less often *jutsu* 'written by' 述) after his name. Watanabe's exact role in the production of this fascicle is unclear.

reducing the number of products needed for the numerical solution to a difficult equation. But [1] has some other noteworthy features. For instance, handwritten notes at the top of two pages call attention to the equivalence of two problems in [1] and in [2], the latter of [2] which was published in 1835, after both Aida and Watanabe had died. This shows that [1] was actively used for some years thereafter. The notes were probably added by Watanabe's disciple Sakuma Masakiyo or one of his descendants, for it was his family that made the foundational donation of Aida's manuscripts to the *wasan* collection of the Yamagata University Library in the Meiji period. We shall point out other historically informative features of the text as we examine the problems it treats in order.

2. EASY ILLUSTRATIVE PROBLEM

Aida introduces [1] by asking, what is the diameter d of the incircle of an isosceles triangle with base $b = 6$ and legs $s = 5$?

From a modern perspective, if h_b is the altitude on the base, there is a right triangle with hypotenuses s with legs equal half the base and to h_b, so $h_b = \sqrt{25 - 9} = 4$, and the area of the triangle is $\frac{1}{2}h_b b = 12$. Its semiperimeter is 8, so if r is its inradius, we immediately have $8r = 12$, or $2r = 3$, which is d.

Aida says that a beginner would calculate $\sqrt{4s^2 - b^2} = 2h_b$ and then $\frac{2h_b b}{2s+b} = d$. I.e. $\sqrt{100 - 36} = 8$ and $\frac{8 \cdot 6}{10+6} = \frac{48}{16} = 3$. But, he notes, $4s^2 - b^2 = (2s + b)(2s - b) = (2h_b)^2$.[3] Hence $2h_b b = b\sqrt{2s + b}\sqrt{2s - b}$ and $\frac{2h_b b}{2s+b} = \frac{b\sqrt{2s-b}}{\sqrt{2s+b}} = d$. I.e. $10 - 6 = 4$ and $6\sqrt{\frac{4}{4+12}} = 6 \cdot \frac{1}{2} = 3$.

Aida's point is evidently that $b\sqrt{\frac{2s-b}{2s+b}}$ is easier to evaluate than $\frac{b\sqrt{4s^2-b^2}}{2s+b}$. Both expressions require extracting a square root and doing a division, but Aida's version requires only two multiplications ($2s$ and $b\sqrt{\frac{2s-b}{2s+b}}$) whereas the beginner's requires five (s^2, b^2, $4s^2, b\sqrt{4s^2 - b^2}$, and $2s$). Since geometry problems were often posed numerically and the Edo period mathematicians who posed and solved them (later known as *wasanka*) did not use logarithms,

---

[3] Aida and his contemporaries usually referred to diameters rather than to radii, so the theorem we would write as $A = \frac{1}{2}rp$, for a triangle of area A and perimeter p, was, for them, $4A = dp$. In this case, $4A = 2h_b b$, $p = 2s + b$; likewise, the relevant Pythagorean relation $s^2 - \frac{b^2}{4} = h_b^2$ was $4s^2 - b^2 = 4h_b^2$.

there was a practical advantage in minimizing the number of multiplications required to get a solution.

3. A HARDER PROBLEM

Given five circles as shown in Figure 1 with diameters $b = 27$ and $c = 12$, what is diameter a?

Aida starts by observing that $s = \sqrt{bc}$ (by a well-known *wasan* theorem), $t = \frac{1}{2}(a+b)$, $2u = 2\sqrt{bc} + c$, $v = w - \frac{1}{2}(b-c)$, and $w = \frac{1}{2}\sqrt{a(a+2c)}$ (from the right triangle with dashed hypotenuse). Using $t^2 = u^2 + v^2$, these definitions lead to $a = \frac{c[b^2 + c\sqrt{bc} + 3b(c+\sqrt{bc})]}{4b(b-c)}$. For the given values of b and c, this yields $a = 25$, as Aida says, but involves at least nine products (and one quotient).

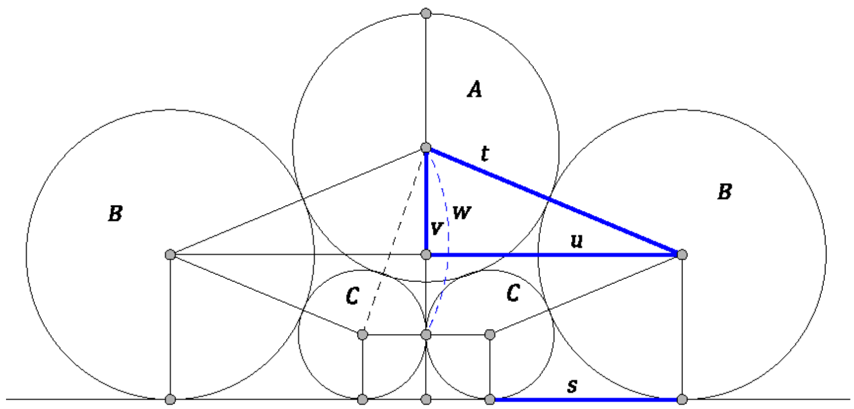

Figure 1.

To obtain the result with fewer products, Aida writes (we silently correct some obvious writing errors)

$$a^2 + 2ac + 2c\sqrt{a(a+2c)} - 2b\sqrt{a(a+2c)} + c^2 - 2bc + b^2 = 4v^2$$
$$4bc + 4c\sqrt{bc} + c^2 = 4u^2$$
$$a^2 + 2ab + b^2 = 4t^2.$$

Using these, he rewrites $4u^2 + 4v^2 - 4t^2 = 0$ in the form

$$2ac + 2c\sqrt{a(a+2c)} - 2b\sqrt{a(a+2c)} + 2c^2 + 2bc + 4c\sqrt{bc} - 2ab = 0,$$

which he divides through by 2 and rearranges to get

$$-a(b-c)-(b-c)\sqrt{a(a+2c)}+c\left(\sqrt{b}+\sqrt{c}\right)^2=0.$$

He now factors $b-c$ as $\left(\sqrt{b}+\sqrt{c}\right)\left(\sqrt{b}-\sqrt{c}\right)$ and divides the equation through by $\left(\sqrt{b}+\sqrt{c}\right)$, rearranges again, and squares both sides:

$$-a\left(\sqrt{b}-\sqrt{c}\right)-\left(\sqrt{b}-\sqrt{c}\right)\sqrt{a^2+2ac}+c\left(\sqrt{b}+\sqrt{c}\right)=0$$
$$-a\left(\sqrt{b}-\sqrt{c}\right)+c\left(\sqrt{b}+\sqrt{c}\right)=\left(\sqrt{b}-\sqrt{c}\right)\sqrt{a^2+2ac}$$
$$a^2\left(\sqrt{b}-\sqrt{c}\right)^2-2ac\left(\sqrt{b}-\sqrt{c}\right)\left(\sqrt{b}+\sqrt{c}\right)+c^2\left(\sqrt{b}+\sqrt{c}\right)^2=\left(\sqrt{b}-\sqrt{c}\right)^2(a^2+2ac).$$

From here (adding some steps that Aida left out),

$$-2ac\left(\sqrt{b}-\sqrt{c}\right)\left(\sqrt{b}+\sqrt{c}\right)+c^2\left(\sqrt{b}+\sqrt{c}\right)^2=2ac\left(\sqrt{b}-\sqrt{c}\right)^2$$
$$-2ac\left(\sqrt{b}-\sqrt{c}\right)\left(\sqrt{b}+\sqrt{c}\right)-2ac\left(\sqrt{b}-\sqrt{c}\right)^2+c^2\left(\sqrt{b}+\sqrt{c}\right)^2=0$$
$$-2ac\left(\sqrt{b}-\sqrt{c}\right)(\sqrt{b}+\sqrt{c}+\sqrt{b}-\sqrt{c})+c^2\left(\sqrt{b}+\sqrt{c}\right)^2=0$$
$$-4ac\sqrt{b}\left(\sqrt{b}-\sqrt{c}\right)+c^2\left(\sqrt{b}+\sqrt{c}\right)^2=0$$
$$-4a\sqrt{b}\left(\sqrt{b}-\sqrt{c}\right)+c\left(\sqrt{b}+\sqrt{c}\right)^2=0$$
$$-4a\left(b-\sqrt{bc}\right)+c\left(\sqrt{b}+\sqrt{c}\right)^2=0.$$

This leads to $a=\frac{(\sqrt{b}+\sqrt{c})^2c}{4(\sqrt{bc}-b)}=\frac{(b+c+2\sqrt{bc})c}{4(\sqrt{bc}-b)}$, which requires just four products (and one division).

4. AN ADVANCED PROBLEM

Given a regular pentagon $ABCDE$ and triangle $AB'E'$ with $B'E'$ passing through C as shown in Figure 2, if $AE'=2.73$ and $AB'=2.50$, what is the length s the pentagon's sides?

It is well known that $a=s\frac{1+\sqrt{5}}{2}=s\phi$, and, by construction, we have $j=s\frac{h}{AE'}$ and $k=s\frac{h}{AB'}$. For convenience, we shall call AB' and AE' η and κ, respectively, and let $m=2\phi$.

From the figure, $c=h-j$ and $b=h-k$. Looking at right triangles, $d^2=s^2-b^2$, $g^2=s^2-j^2$, $i^2=s^2-k^2$; so too $e^2=a^2-c^2$, $f^2=a^2-h^2$.

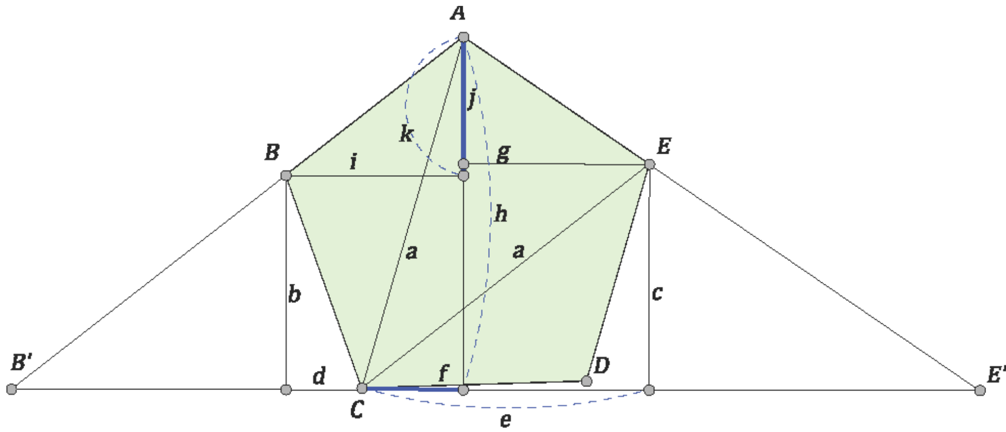

Figure 2.

Because $e = f + g$, $e^2 - f^2 - g^2 = 2fg$. Therefore, $(e^2 - f^2 - g^2)^2 = 4(a^2 - h^2)(s^2 - j^2) = 4a^2s^2 - 4h^2s^2 - 4a^2j^2 + 4h^2j^2$. Calculating more directly, $e^2 - f^2 - g^2 = (a^2 - c^2) - (a^2 - h^2) - (s^2 - j^2) = -c^2 + h^2 - s^2 + j^2$. Substituting $(h - c)^2$ for j^2, this is $2h^2 - s^2 - 2ch = 2hj - s^2$. So, squaring, $(e^2 - f^2 - g^2)^2 = s^4 - 4hjs^2 + 4h^2j^2$. Setting this equal to the expression for $4f^2g^2$,

$$s^4 - 4hjs^2 - 4a^2s^2 + 4h^2s^2 + 4a^2j^2 = 0. \quad (1)$$

Likewise, because $i = d + f$, $i^2 - d^2 - f^2 = 2df$. Therefore, $(i^2 - d^2 - f^2)^2 = 4(s^2 - b^2)(a^2 - h^2) = 4a^2s^2 - 4a^2b^2 - 4h^2s^2 + 4b^2h^2$. Calculating more directly, $i^2 - d^2 - f^2 = (s^2 - k^2) - (s^2 - b^2) - (a^2 - h^2) = -k^2 + b^2 - a^2 + h^2$. Substituting $-(h - b)^2$ for $-k^2$, this is $-a^2 - 2bh$. So, squaring, $(i^2 - d^2 - f^2)^2 = a^4 - 4a^2bh + 4b^2h^2$. Setting this equal to the expression for $4d^2f^2$,

$$a^4 - 4a^2bh - 4a^2s^2 + 4a^2b^2 + 4h^2s^2 = 0. \quad (2)$$

We can simplify (1) and (2) by substituting $ms/2$ for a and dividing through by s^2. For (1), we finish by substituting $\frac{sh}{\eta}$ for j:

$$s^2\eta^2 - 4h^2s\eta - 4s^2m^2\eta^2 + 4h^2\eta^2 + h^2m^2s^2 = 0. \quad (3)$$

For (2), we first substitute $h - k$ for b, then sh/κ for k, and finally multiply through by 16:

$$m^4s^2\kappa^2 - 16h^2s\kappa + 16h^2m^2s^2 + 16m^2s^2\kappa^2 + 64h^2\kappa^2 = 0. \quad (4)$$

Now, rearranging (3) and (4) as linear equations in h^2, we get, respectively,

$$s^2(\eta^2 - 4m^2\eta^2) + h^2(4\eta^2 + m^2s^2 - 4s\eta) = 0$$

and

$$s^2(m^4\kappa^2 + 16m^2\kappa^2) + h^2(64\kappa^2 - 16s\kappa + 16m^2s^2) = 0.$$

Both equations hold if and only if the determinant of their coefficients is zero:

$$\begin{vmatrix} \eta^2 - 4m^2\eta^2 & 4\eta^2 + m^2s^2 - 4s\eta \\ m^4\kappa^2 + 16m^2\kappa^2 & 64\kappa^2 - 16s\kappa + 16m^2s^2 \end{vmatrix} = 0,$$

This leads to a quadratic equation in s:

$$64\eta^2\kappa^2 - 4m^4\eta^2\kappa^2 + s(-16m^2\eta^2\kappa + 16m^4\eta^2\kappa - 64m^2\eta\kappa^2 + 4m^4\eta\kappa^2) + s^2(16m^2\eta^2 - 16m^4\eta^2 + 16m^4\kappa^2 - m^6\kappa^2) = 0, \qquad (5)$$

which is Aida's basic solution. A program such as Mathematica quickly finds the roots 0.98666 … (Aida gives $s = 0.98659$) and 2.20009 … for $\eta = 2.73$ and $\kappa = 2.5$. Aida, however, had to solve (5) "by hand" and wants to illustrate techniques for simplifying equations like (5) algebraically before finding s. He therefore first divides (5) through by $4\eta^2\kappa^2$, in effect redefining its unknown variable as $t = \frac{s}{2\eta\kappa}$:

$$16 - m^4 + t(-8m^2\eta + 8m^4\eta - 32m^2\kappa^2 + 2m^4\kappa) + t^2(16m^2\eta^2 - 16m^4\eta^2 + 16m^4\kappa^2 - m^6\kappa^2) = 0.$$

Next, he defines $n = m^2 - 1$ and $p = 16 - m^2$ and uses these to rewrite the equation as

$$p - m^2n + t(-2p\kappa + 8m^2n\kappa^2) - t^2(m^4p\eta - 16m^2n\eta^2\ ) = 0.$$

Multiplying through by n, this is

$$np - m^2n^2 + t(-2np\kappa + 8m^2n^2\kappa^2) - t^2(m^4np\eta - 16m^2n^2\eta^2\ ) = 0.$$

Aida implicitly treats this as

$$np - (2np\kappa)t - (m^4np\kappa^2)t^2 = m^2n^2 - (8m^2n^2\eta)t + (16m^2n^2\eta^2\ )t^2,$$

in which both sides are perfect squares:

$$\left[\left(m^2\sqrt{np}\kappa\right)t-\sqrt{np}\right]^2=[(4mn\eta)t-mn]^2.$$

Taking roots and rearranging,

$$t\left(m^2\sqrt{np}\kappa+4mn\eta\right)-\sqrt{np}-mn=0,$$

or, substituting $\frac{s}{2\eta\kappa}$ for t and clearing the denominator,

$$s\left(m^2\sqrt{np}\kappa+4mn\eta\right)-2\sqrt{np}\eta\kappa-2mn\eta\kappa=0.$$

Aida now observes that $\sqrt{np}=m\sqrt{5}$ and writes

$$s\left(m^2\sqrt{5}\kappa+4n\eta\right)-2\sqrt{5}\eta\kappa-2n\eta\kappa=0.$$

Substituting $2(3+\sqrt{5})$ for m^2 and dividing through by 2, this becomes

$$s\left[\left(3+\sqrt{5}\right)\sqrt{5}\kappa+4n\eta\right]-\sqrt{5}\eta\kappa-n\eta\kappa=0,$$

or, since $n=5+2\sqrt{5}$,

$$s\left[\left(3+\sqrt{5}\right)\kappa+2\left(2+\sqrt{5}\right)\eta\right]-(3+\sqrt{5})\eta\kappa=0. \tag{6}$$

That is, $s=\frac{(3+\sqrt{5})\eta\kappa}{(3+\sqrt{5})\kappa+2(2+\sqrt{5})\eta}$.[4] This minimizes multiplications, but uses $\sqrt{5}$ four times, so Aida explains how to eliminate three of these by showing that $\frac{(3+\sqrt{5})\eta\kappa}{(3+\sqrt{5})\kappa+2(2+\sqrt{5})\eta}=\frac{2\eta\kappa}{\eta(\sqrt{5}+1)+2\kappa}$ (with four products), which he reduces to $\frac{\eta\kappa}{\eta(\sqrt{5}+1)/2+\kappa}$ (with three, including the division by 2).

Finally, Aida turns his attention to finding h. From a modern perspective, we note that $\angle B'AE'$ has measure $\frac{3\pi}{5}=108\,°$ and use the Law of Cosines to find the third side θ; then, using Heron's formula to find the area of the triangle, we multiply it by $\frac{2}{\theta}$ to get h. For Aida, a trigonometric solution was out of bounds, so he substitutes $\frac{2\eta\kappa}{\eta(\sqrt{5}+1)+2\kappa}$ for s in (3) and solves for h^2. He concludes with two expressions for its positive square root.

---

[4] Aida writes this correctly in his algebraic notation, but his narrative description of it, which comes immediately after, is garbled and is equivalent to $s=\frac{(3+\sqrt{5})\eta\kappa}{(3+\sqrt{5})\eta+(2+\sqrt{5})\kappa}$.

5. THE DESCARTES CIRCLE THEOREM

The importance to Aida of simplifying numerical calculations is clear from the excursions into algebra, such as the foregoing, that fill this book, but, for the modern reader, the information it holds about what Aida and his disciples took for granted is perhaps even more valuable. Such common knowledge evidently included the Descartes Circle Theorem!

For convenience, we shall call a set of n circles, $n \geq 3$, in which each touches two others externally, a *closed chain*. If one tangency of a closed chain is removed, the circles form an *open chain*.[5] Problem 4 asks for the diameter of the circle (O) internally tangent to three circles $(A), (C), (D)$ of known diameters in a closed chain. Problem 5 asks for the diameter of the circle (B) externally tangent to all three given circles. These two problems correspond to the two cases of the Descartes Circle Theorem. A modern solution of problem 5, given in [3: 284–93], is tantamount to a proof of the theorem and yields a formula very similar to Aida's solution. Significantly, Aida works out the derivation of the solution of problem 4 in detail but simply describes the solution of problem 5 as a variation of it.

Aida had proven the Descartes Circle theorem by 1810 [6: 6–7] and he was by no means the first *wasanka* to do so; moreover, the two marginal notes mentioned earlier, which refer to the texts of problems 4 and 5, state that these problems are essentially the same as problems 64 and 65 in the second fascicle of [2] except for the given numerical values. [2] is one of many works written by students of Hasegawa Hiroshi (1782–1838) 長谷川寛 as described in [4] .[6] Hasegawa's mentors traced their academic lineage back to Seki Takakazu (1642–1708). The annotated copy of [1] in the Yamagata University Library thus shows not only that it was consulted after Aida's death but also that those who carried on Aida's work were well aware of the accomplishments of *wasanka* outside their own school.

Problem 6 also concerns the five circles $(O), (A), (B), (C), (D)$ of the foregoing problems but with (O) replaced by a straight line tangent to $(A), (C), (D)$, which now form an open chain (Figure 3). Given diameters a, c, d, what is b? That this problem is included at this point in [1] suggests that Aida had an intuitive understanding of curvature if not also of "dynamic" geometry: as the diameter of a circle increases without limit, its circumference becomes a straight line.

---

[5] Constructing closed and open chains of three circles is easy. To construct a circle external to two $(A)r$ and (B) at a point P on (B), one constructs $(P)r$; denoting $(P) \cap PB$ as Q, the perpendicular bisector of AQ meets BP extended in C such that $(C)P$ is externally tangent to both (A) and (B). If (A) and (B) happen to be externally tangent to one another, then (C) completes a closed chain.

[6] These pages give not only a list books by other students of Hasegawa but also an explanation of the method he called *kyokugyō* and how it was eventually found to be unsound.

By a well-known *wasan* theorem, $PQ + QR = \sqrt{ad} + \sqrt{cd}$ in Figure 3, so $AC^2 = d(\sqrt{a} + \sqrt{c})^2 + \frac{(a-c)^2}{4}$. To get another expression for AC^2, we first apply the Law of Cosines (in the form the *wasanka* knew it) to $\triangle ABD$: $AB^2 + BD^2 - AD^2 = 2BT \cdot BD$.[7] Since $AB^2 = \frac{(a+b)^2}{4}$, $BD^2 = \frac{(b+d)^2}{4}$, and $AD^2 = ad + \frac{(a-d)^2}{4}$, this implies $\frac{b^2}{4BD} + \frac{bd}{4BD} + \frac{ab}{4BD} - \frac{ad}{4BD} = BT$. Likewise, if we start with $\triangle ACD$, $\frac{b^2}{4BD} + \frac{bd}{4BD} + \frac{bc}{4BD} - \frac{cd}{4BD} = BS$. Hence $BT - BS = ST = \frac{(b-d)(a-c)}{4BD}$.

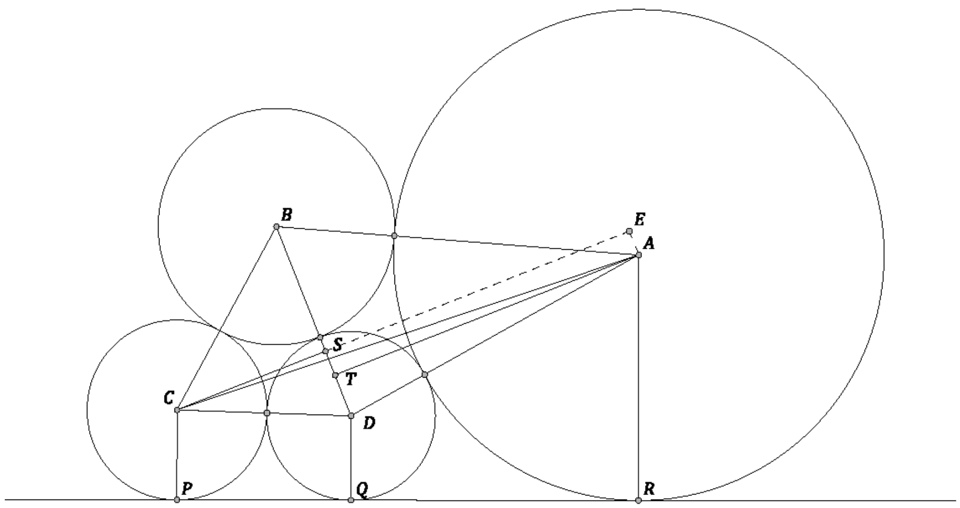

Figure 3.

Now $AB^2 - BT^2 = AT$ and $BC^2 - BS^2 = CS^2$, so $AT^2 = \frac{abd(a+b+d)}{4BD^2}$ and $CS^2 = \frac{bcd(b+c+d)}{4BD^2}$. Temporarily denoting $a(a+b+d)$ and $c(b+c+d)$ as e and f, respectively, these become $AT^2 = \frac{bde}{4BD^2}$ and $CS^2 = \frac{bdf}{4BD^2}$, or $AU = \frac{\sqrt{bde}}{2BD}$ and $CS = \frac{\sqrt{bdf}}{2BD}$. Squaring their sum and adding ST^2 gives the alternative expression for AC^2 we seek (N.B. $\triangle CEA$ is a right triangle): $\frac{bde}{4BD^2} + 2\frac{bd\sqrt{ef}}{4BD^2} + \frac{bdf}{4BD^2} + \frac{(b-d)^2(a-c)^2}{4BD^2}$.

Subtracting from this our first expression for AC^2, we obtain $\frac{bde}{(b+d)^2} + 2\frac{bd\sqrt{ef}}{(b+d)^2} + \frac{bdf}{(b+d)^2} + \frac{(b-d)^2(a-c)^2}{4(b+d)^2} - d(\sqrt{a} + \sqrt{c})^2 - \frac{(a-c)^2}{4} = 0$. Multiplying through by $(b+d)^2$, we note that the

[7] Aida neglects to write the minus sign.

fourth and sixth terms reduce to $bd(a-c)^2$; hence, all five terms have a factor of d, and our equation becomes $be+bf-b(a-c)^2-\left(\sqrt{a}+\sqrt{c}\right)^2(b+d)^2=-2b\sqrt{ef}$.

Aida proceeds to square both sides of this equation, replace e and f, and simplify. Omitting the steps he took, suffice to say that the equation can be written as $\left(\sqrt{a}+\sqrt{c}\right)(b+d)\left[cd^2+2\sqrt{ac}d(2b+d)+a(d^2-4bc)\right]=0$, the first two factors of which are positive; hence $cd^2+2\sqrt{ac}d(2b+d)+a(d^2-4bc)=0$, with the solution $b=\frac{\left(\sqrt{a}+\sqrt{c}\right)^2d^2}{4ac-4\sqrt{ac}d}$.

Evaluating this expression numerically requires forming at least six products (two in the numerator and four in the denominator), finding at least two square roots, and performing one division. Aida gives three expressions for b, each with fewer products: (1) $c\left(\sqrt{\frac{a}{c}}d+d\right)^2$; (2) $\frac{a-\sqrt{\frac{a}{c}}d}{\left(\frac{\sqrt{\frac{a}{c}}d+d}{2}\right)^2}$; and, (3) $\frac{4(\sqrt{ac}-d)}{2d^2\left(\frac{\sqrt{ac}}{a+c}+2\right)}$. All these require finding at least one square root and performing at least one division; (1) requires just three multiplications, (2) requires two but an extra division, and (3) requires five and an extra division.

Aida doesn't discuss how to construct B given the open chain of $(A),(C),(D)$. He apparently knew that Figure 3 could always be constructed, formulated the equations it implied, and concentrated on finding diameter b. Given b as well as any two of the circles (A) , (C), and (D) in an open chain with a common external tangent, it is not hard to find B such that (B) touches all three externally.

Problem 7 is an analogous problem with spheres instead of circles. Given a chain—this time closed, not open—of three spheres $(B),(C),(D)$ with diameters b,c,d, all tangent to a common plane, there is a fourth sphere (A) with no point on the plane externally tangent to all three. Given the perpendicular to the plane through A, let the distance along it from the plane to the farthest point on (A) be h: what is diameter a in terms of b,c,d,h?

6. THE LAST FOUR PROBLEMS

Returning to two dimensions, problem 8 concerns two closed chains of three circles that have two members in common. That is, given two externally tangent circles $(A)\frac{a}{2}$ and $(B)\frac{b}{2}$, one adds circles $(C)\frac{c}{2}$ and $(D)\frac{d}{2}$ externally tangent to both on opposite sides of AB. What is the distance CD? The solution, $CD = \frac{\left[\frac{1}{2}(a-b)(c-d)+2ab\sqrt{c(a+b+c)}\sqrt{d(a+b+d)}\right]^2}{a+b}$, is explicitly identified in the introduction to problem 9, as a lemma.

Problems 9 and 10 concern a closed chain of four circles, to each of a which a fifth circle is, respectively, externally or internally tangent. In both problems, the diameters of all the circles are given except for one in the chain, which must be found. The solution of 9 is given in detail; solution 10, referring back to 9, is highly abbreviated. Finally, problem 11 concerns four spheres of given diameters in a closed chain. Since there is a plane tangent to any three of these, the configuration is analogous to that of Problem 7 (three of the spheres touching a common plane, as in Figure 4, with the fourth tangent to the other three): what is the diameter of a fifth sphere (not shown) internally tangent to all four?

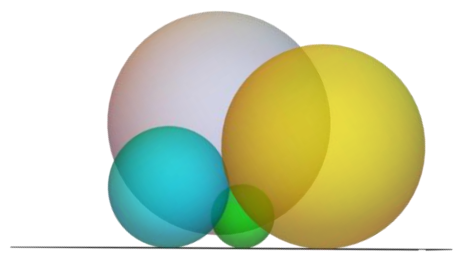

Figure 4.

The solution is too long to paraphrase in this article; in any case, from a historical perspective, it seems less important than the fact that the problem configuration is almost (but not quite) Soddy's Hexlet of spheres (see [5]). It has long been known that the *wasanka* had discovered the hexlet independently by 1822 ([3: 288]), but [1] suggests that, though Aida may have been unaware of it, he was certainly close to finding it even earlier.

References

[1] Yasuaki AIDA 会田安明. WATANABE Jiemon Kazu 渡辺治右衛門一 (?ed.). n.d. *Sanpō tai kajō no daishū* (A collection of problems with calculations that entail an excessive number of multiplications) 算法帯過乗之題集. 1 fascicle.

[2] Jūshichirō Giichi AKITA 秋田十七郎義一. 1835. *Kyokugyō shinan* (A guide to the method of transforming geometric figures) 極形指南. 3 fascicles. Edo [Tōkyō]: Nishimiya Yahei 西宮弥兵衛.

[3] Hidetoshi FUKAGAWA and Tony ROTHMAN. 2008. *Sacred Mathematics: Japanese Temple Geometry*. Princeton & Oxford: Princeton University Press.

[4] Kazuo SHIMODAIRA. 2011 [1972]. *Nihonjin no sūgaku: wasan* (The mathematics of the Japanese: *wasan*) 日本人の数学　和算. Tōkyō: Kōdansha.

[5] Frederick SODDY. 1937. The bowl of integers and the hexlet, *Nature*, vol. 139 no. 3506 pp. 77–79.

[6] J. Marshall UNGER. 2024. Notes on the Six-Segment Theorem. *Sangaku Journal of Mathematics*, vol. 1 no. 8 pp. 1–8.