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Two Beautiful Sangaku Theorems

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Abstract. We will explore two elegant Sangaku Theorems: one associated with cyclic quadrilaterals and another related to bicentric quadrilaterals and establish a connection to a known Sangaku Theorem.

Keywords. Sangaku, Curvilinear triangle, Stewart's Theorem, Apollonian.

Mathematics Subject Classification (2010). 51M04.

1. Introduction

We explore two beautiful Sangaku Theorems: one focusing on cyclic quadrilaterals and the other on bicentric quadrilaterals and establish a connection to a known Sangaku Theorem. First, we will explore the following Theorem associated with cyclic quadrilaterals.

Theorem 1.1. *Let $ABCD$ be a convex cyclic quadrilateral where $P = AC \cap BD$. (See Figure 1) Let $O_i(r_i)$ $(i = 1, 2, 3, 4)$ be incircles of triangles APB, BPC, CPD, and DPA, respectively, and $W_i(w_i)$ $(i = 1, 2, 3, 4)$ be incircles of curvilinear triangles APB, BPC, CPD and DPA, respectively, then*

$$\frac{1}{r_i} + \frac{1}{w_{i+2}}$$

is constant. If suffixes are greater than four we will take modulo 4.
More explicitly, we may write

$$\frac{1}{r_1} + \frac{1}{w_3} = \frac{1}{r_2} + \frac{1}{w_4} = \frac{1}{r_3} + \frac{1}{w_1} = \frac{1}{r_4} + \frac{1}{w_2}.$$

Here the term *'curvilinear triangle'* means a triangle whose two sides are straight lines and one side is an arc of a circle.

Throughout we will use $O_i(r_i)$ for incircles of triangles and $W_i(w_i)$ for incircles of curvilinear triangles as stated in **Theorem 1.1**.

To prove the above Theorem we need the following two lemmas.

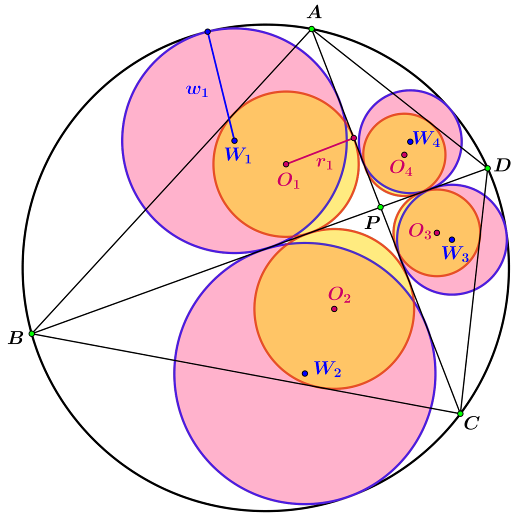

FIGURE 1

Lemma 1. *Let $ABCD$ be a convex cyclic quadrilateral where $P = AC \cap BD$. (See Figure 2). Let $O_1(r_1)$ and $O_4(r_4)$ be incircles of triangles APB and DPA, respectively, and $W_2(w_2)$ and $W_3(w_3)$ be incircles of curvilinear triangles BPC, and CPD, respectively, then*

$$\frac{1}{r_1} + \frac{1}{w_3} = \frac{1}{r_4} + \frac{1}{w_2}.$$

Proof. The above Lemma is a rephrased version of **Theorem 3.2** proved by Dr. Stanley Rabinowitz in [[4], pp. 14-15]. □

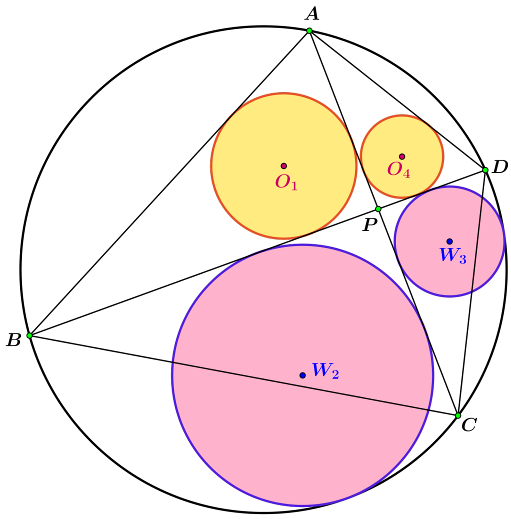

FIGURE 2

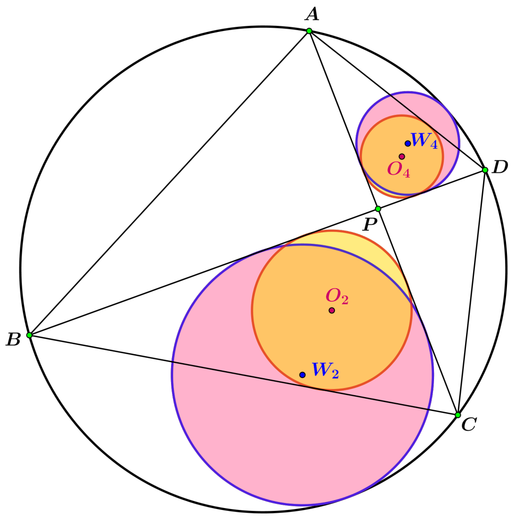

FIGURE 3

Lemma 2. *Let $ABCD$ be a convex cyclic quadrilateral where $P = AC \cap BD$. (See Figure 3). Let $O_2(r_2)$ and $O_4(r_4)$ be incircles of triangles BPC and DPA, respectively, and $W_2(w_2)$ and $W_4(w_4)$ be incircles of curvilinear triangles BPC and DPA, respectively, then*

$$\frac{1}{r_2} + \frac{1}{w_4} = \frac{1}{r_4} + \frac{1}{w_2}.$$

Proof. The above Lemma is a rephrased version of **Theorem 3.3** proved by Dr. Stanley Rabinowitz in [[4], pp. 16]. □

We are about to uncover that **Theorem 1.1** is a consequence of **Lemma 1** and **Lemma 2**. A proof of **Theorem 1.1** is given below.

Proof. Applying **Lemma 1** for pairs $\{(r_1, w_3), (r_4, w_2)\}$ and $\{(r_2, w_4), (r_3, w_1)\}$, we obtain

$$\frac{1}{r_1} + \frac{1}{w_3} = \frac{1}{r_4} + \frac{1}{w_2} \quad \text{and} \quad \frac{1}{r_2} + \frac{1}{w_4} = \frac{1}{r_3} + \frac{1}{w_1}. \tag{1}$$

Applying **Lemma 2** for pairs $\{(r_1, w_3), (r_3, w_1)\}$ and $\{(r_2, w_4), (r_4, w_2)\}$, we obtain

$$\frac{1}{r_1} + \frac{1}{w_3} = \frac{1}{r_3} + \frac{1}{w_1} \quad \text{and} \quad \frac{1}{r_2} + \frac{1}{w_4} = \frac{1}{r_4} + \frac{1}{w_2}. \tag{2}$$

Combining equations (1) and (2), we obtain

$$\frac{1}{r_1} + \frac{1}{w_3} = \frac{1}{r_2} + \frac{1}{w_4} = \frac{1}{r_3} + \frac{1}{w_1} = \frac{1}{r_4} + \frac{1}{w_2}.$$ □

2. THE SECOND THEOREM

Now, we will offer proof of the following Theorem related to bicentric quadrilaterals, which was proposed by Mr. Keita Miyamoto [3].

Theorem 2.1. *Let $ABCD$ be a convex bicentric quadrilateral with circumcircle $O(R)$ where $P = AC \cap BD$. (See Figure 4). Let $W_i(w_i)$ $(i = 1, 2, 3, 4)$ be incircles of curvilinear triangles APB, BPC, CPD, and DPA, respectively, then*

$$\frac{1}{w_1} + \frac{1}{w_3} = \frac{1}{w_2} + \frac{1}{w_4},$$

and there exists a circle externally tangent to (W_1), (W_2), (W_3), and (W_4).

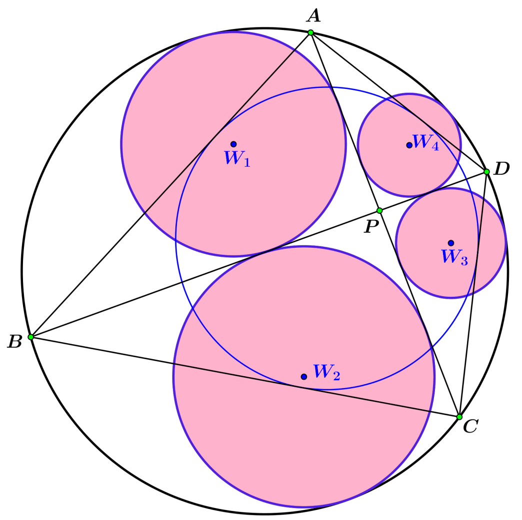

FIGURE 4

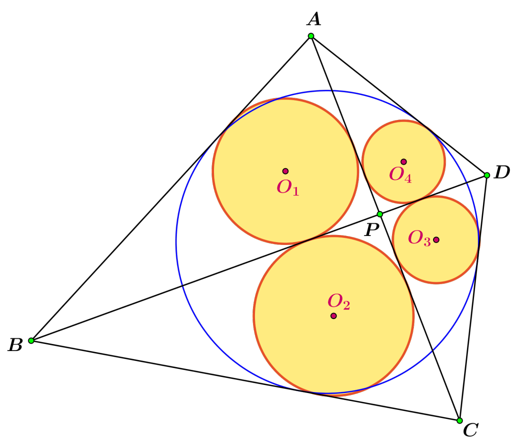

FIGURE 5

To prove the above Theorem, we need **Theorem 1.1** and the following lemma.

Lemma 3. *Let $ABCD$ be a convex quadrilateral where $P = AC \cap BD$. (See Figure 5). If $O_i(r_i)$ $(i = 1, 2, 3, 4)$ be incircles of triangles APB, BPC, CPD and DPA, respectively. Then convex quadrilateral $ABCD$ is tangential if and only if*

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}.$$

Proof. A proof can be found in [[1], pp. 657–658]. □

Proof of Theorem 2.1.

A bicentric quadrilateral is both cyclic and tangential. So, for convex bicentric quadrilateral $ABCD$, **Theorem 1.1** and **Lemma 3** both hold.

From **Theorem 1.1**, we can write

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{w_1}+\frac{1}{w_3}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{w_2}+\frac{1}{w_4}.$$

Lemma 3 gives,

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}.$$

The above two relations give

$$\frac{1}{w_1}+\frac{1}{w_3}=\frac{1}{w_2}+\frac{1}{w_4}.$$

This proves the first part of the **Theorem 2.1**. To prove the second part, let us assume that $\angle APW_1=\angle BPW_1=\angle CPW_3=\angle DPW_3=\alpha$ (see Figure 6) and denote circumcircle $\odot(ABC)$ by $O(R)$. Then $OW_1=R-w_1$, $OW_3=R-w_3$. Then $PW_1=kw_1$ and $PW_3=kw_3$ where $k=\operatorname{cosec}\alpha$. Applying Stewart's Theorem on triangle OW_1W_3, we obtain

$$(R-w_1)^2\cdot kw_3+(R-w_3)^2\cdot kw_1=k(w_1+w_3)(OP^2+k^2w_1w_3)$$

$$\Rightarrow R^2-OP^2=\frac{4Rw_1w_3}{w_1+w_3}+(k^2-1)\cdot w_1w_3=\frac{4Rw_1w_3}{w_1+w_3}+w_1w_3\cot^2\alpha. \quad (3)$$

Since $\angle BPW_2=\angle CPW_2=\angle DPW_4=\angle APW_4=90^\circ-\alpha$; $OW_2=R-w_2$ and $OW_4=R-w_4$, applying Stewart's Theorem on triangle OW_2W_4, we obtain

$$R^2-OP^2=\frac{4Rw_2w_4}{w_2+w_4}+w_2w_4\tan^2\alpha. \quad (4)$$

Since

$$\frac{1}{w_1}+\frac{1}{w_3}=\frac{1}{w_2}+\frac{1}{w_4}\ \Rightarrow\ \frac{w_1+w_3}{w_1w_3}=\frac{w_2+w_4}{w_2w_4}, \quad (5)$$

from (3) and (4), we get

$$w_1w_3\cot^2\alpha=w_2w_4\tan^2\alpha. \quad (6)$$

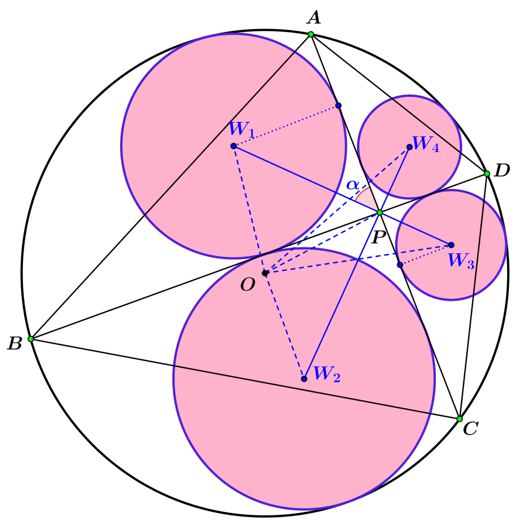

Figure 6

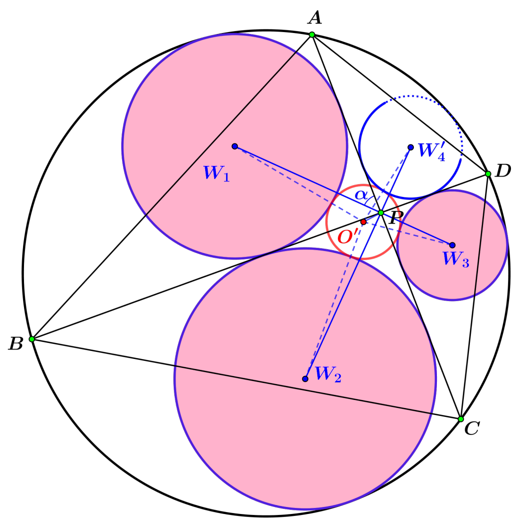

Figure 7

Now assume there exists a circle $O'(r)$ which is externally tangent to (W_1), (W_2), and (W_3). We know that, by Apollonian construction, such a circle exists. We further assume that there exists a circle $W_4'(w_4')$ which is tangent to PD, PA, and externally tangent to $O'(r)$ (see Figure 7). Again, by Apollonian construction, such a circle exists. If we can prove (W_4') and (W_4) are identical then the result will follow.

Considering circles $O'(r)$, $W_1(w_1)$, $W_3(w_3)$, and applying Stewart's Theorem on triangle $O'W_1W_3$, we obtain

$$r^2 - O'P^2 = -\frac{4rw_1w_3}{w_1+w_3} + w_1w_3\cot^2\alpha, \tag{7}$$

and considering circles $O'(r)$, $W_2(w_2)$, $W_4'(w_4')$, and applying Stewart's Theorem on triangle $O'W_2W_4'$, we obtain

$$r^2 - O'P^2 = -\frac{4rw_2w_4'}{w_2+w_4'} + w_2w_4'\tan^2\alpha. \tag{8}$$

Using (5) and (6), we rewrite (7) as

$$r^2 - O'P^2 = -\frac{4rw_2w_4}{w_2+w_4} + w_2w_4\tan^2\alpha. \tag{9}$$

From (8) and (9), we obtain

$$(w_4 - w_4')\left(r^2 - O'P^2 + \frac{4rw_2w_4w_4'}{(w_4+w_2)(w_4'+w_2)}\right) = 0.$$

The first factor gives $w_4 = w_4'$ and the second factor gives

$$r^2 - O'P^2 = -\frac{4rw_2w_4w_4'}{(w_4+w_2)(w_4'+w_2)},$$

which is not possible as $r^2 - O'P^2$ is positive. Therefore, the only possibility is $w_4' = w_4$ and the distance $|O'W_4'| = r + w_4' = r + w_4 = |O'W_4|$. So, circles (W_4') and (W_4) are identical. Therefore, there exists a circle externally tangent to (W_1), (W_2), (W_3), and (W_4).

Corollary 2.1. *From* (5) *and* (6)*, we find*

$$\frac{w_1+w_3}{w_2+w_4} = \frac{w_1w_3}{w_2w_4} = tan^4\alpha.$$

Corollary 2.2. *Let* $W_i(w_i)$ $(i = 1, 2, 3, 4)$ *be circles which are tangent to two intersecting lines* l *and* m*. If these four circles touch a circle* (O') *externally and the relationship stated in* (5) *holds then there exists a circle internally tangent to* (W_i) $(i = 1, 2, 3, 4)$.

Corollary 2.3. *Let* $W_i(w_i)$ $(i = 1, 2, 3, 4)$ *be circles which are tangent to two intersecting lines* l *and* m *such that the angle between them is* 2α *where* α *satisfies* (6)*. If the relationship stated in* (5) *holds then there exist two circles, one internally and the other externally tangent to* (W_i) $(i = 1, 2, 3, 4)$.

3. A connection to a known Sangaku Theorem

Now look at the following Theorem, given as **Problem 1.5.11** in [2].

Theorem 3.1. *Two chords AC and BD divide a circle $O(R)$ into four curvilinear triangles. Let $O_i(w_i)$ $(i = 1, 2, 3, 4)$ be the circles inscribed in these curvilinear triangles. If these four circles all touch a circle $O'(r)$ externally, show that*

$$\frac{1}{w_1} + \frac{1}{w_3} = \frac{1}{w_2} + \frac{1}{w_4}.$$

Proof. The proof of the above Theorem is very similar to the proof of part two of **Theorem 2.1**. Equating (3) and (4), we obtain

$$\frac{4Rw_1w_3}{w_1 + w_3} + w_1w_3 \cot^2 \alpha = \frac{4Rw_2w_4}{w_2 + w_4} + w_2w_4 \tan^2 \alpha.$$

Similarly, we obtain

$$-\frac{4rw_1w_3}{w_1 + w_3} + w_1w_3 \cot^2 \alpha = -\frac{4rw_2w_4}{w_2 + w_4} + w_2w_4 \tan^2 \alpha.$$

Therefore,

$$(R + r)\frac{4Rw_1w_3}{w_1 + w_3} = (R + r)\frac{4Rw_2w_4}{w_2 + w_4},$$

which implies

$$\frac{1}{w_1} + \frac{1}{w_3} = \frac{1}{w_2} + \frac{1}{w_4}.$$ □

This Theorem doesn't assume that quadrilateral $ABCD$ is tangential but we will show that it is indeed the case. From **Theorem 1.1**, we can write

$$\frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{w_1} + \frac{1}{w_3} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{w_2} + \frac{1}{w_4}.$$

Using **Theorem 3.1**, we get

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}.$$

Since **Lemma 3** holds for necessary and sufficient conditions, we conclude that $ABCD$ must be tangential, i.e., $ABCD$ is a bicentric quadrilateral. So the new version of this Theorem may be stated in the following manner:

Theorem 3.2. *Let $ABCD$ be a convex cyclic quadrilateral where $P = AC \cap BD$ and let (W_i) $(i = 1, 2, 3, 4)$ be incircles of curvilinear triangles APB, BPC, CPD, and DPA, respectively. If these four circles all touch a circle (O') externally, then quadrilateral $ABCD$ is bicentric.*

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