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Relationships Between Radii in a Six-Circle Configuration

(Dedicated to Late Dr. Alexander Bogomolny)

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Abstract. We will prove some relationships among six radii on certain six-circle configurations and discuss some related problems.

Keywords. Circle configuration, Stewart's Theorem, Cosine law, Inversion.

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1. Introduction

In this paper, we will prove some relationships among six radii of certain six-circle configurations, such as given in Figure 1, and give possible generalizations and discuss some related problems.

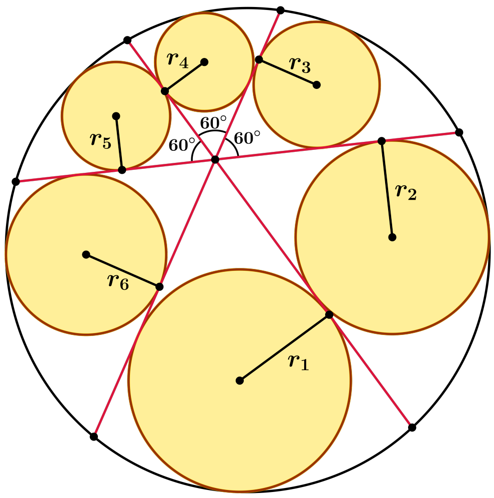

Figure 1.

Theorem 1.1. *Through point P, inside circle $O(R)$ three chords are drawn at $60°$ to each other and they cut the circle $O(R)$ successively in six points-A_i $(i = 1, 2, \ldots, 6)$. If six circles $O_i(r_i)$ $(i = 1, 2, \ldots, 6)$ are inscribed into the curvilinear triangles A_iPA_{i+1} for $i = 1, 2, \ldots, 6$, $(A_7 \equiv A_1)$ (See Figure 1), then*

$$r_1r_3 + r_3r_5 + r_5r_1 = r_2r_4 + r_4r_6 + r_6r_2.$$

This result was proposed by Borislav Mirchev [2] of Bulgaria, ten years ago. No traditional proof is known for this result, only an analytic solution was provided by Leo Giugiuc of Romania, almost ten years ago. Here we offer a traditional proof.

2. PROOF OF THEOREM 1.1

Proof. Join P, O_1; P, O_3; P, O_5; O, P; O, O_1; O, O_3, and O, O_5. Let N_1 be the orthogonal projection of O_1 on A_1A_4. Since $O_1N_1 = r_1$ and $\angle N_1PO_1 = \angle A_1PO_1 = \frac{1}{2}\angle A_1PA_2 = 30°$, therefore, $PO_1 = 2r_1$. Similarly, $PO_3 = 2r_3$ and $PO_5 = 2r_5$. Also, $\angle O_1PO_3 = \angle O_1PO_5 = 120°$, $OO_1 = R - r_1$, $OO_3 = R - r_3$, and $OO_5 = R - r_5$. Let $\angle OPO_1 = \theta$ and $OP = m$. We assume $a = r_1$, $b = r_3$, and $c = r_3$ for easier presentation.

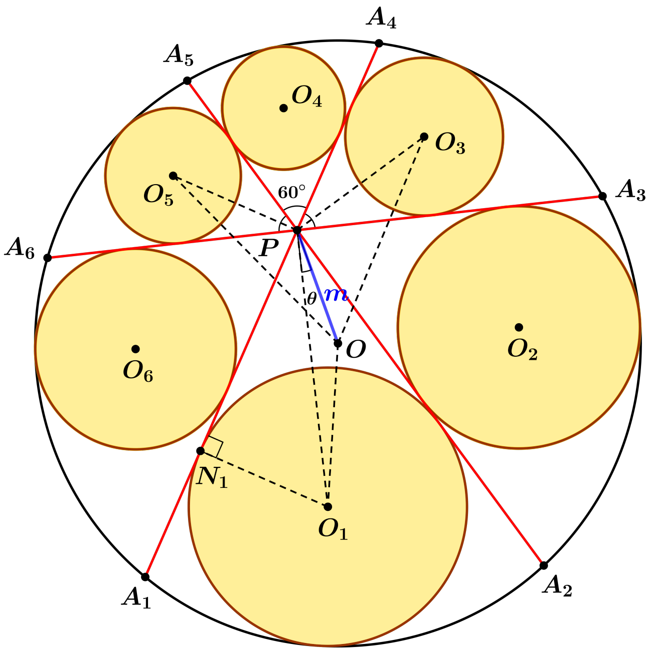

FIGURE 2.

Using cosine law to triangle OPO_1, we get

$$\cos\theta = \frac{m^2 + 4r_1^2 - (R - r_1)^2}{4mr_1} = \frac{m^2 + 4a^2 - (R - a)^2}{4ma}. \tag{1}$$

Since $\angle OPO_3 = 120° - \theta$, using cosine law to triangle OPO_3, we get

$$\cos(120° - \theta) = \frac{m^2 + 4r_3^2 - (R - r_3)^2}{4mr_3} = \frac{m^2 + 4b^2 - (R - b)^2}{4mb}. \tag{2}$$

Since $\angle OPO_5 = 120° + \theta$, using cosine law to triangle OPO_5, we get

$$\cos(120° + \theta) = \frac{m^2 + 4r_5^2 - (R - r_5)^2}{4mr_5} = \frac{m^2 + 4c^2 - (R - c)^2}{4mc}. \tag{3}$$

Using (1), (2), (3), and the following well-known identity

$$\cos\theta + \cos(120^\circ - \theta) + \cos(120^\circ + \theta) = 0,$$

we can easily derive

$$(R^2 - m^2)(ab + bc + ca) = 6Rabc + 3abc(a + b + c). \tag{4}$$

Again, using (1), (2), (3), and the following well-known identity

$$\cos\theta\cos(120^\circ - \theta) + \cos(120^\circ - \theta)\cos(120^\circ + \theta) + \cos(120^\circ + \theta)\cos\theta = -\frac{3}{4},$$

we get

$$\sum_{a,\ b,\ c}\left(\frac{m^2 + 4a^2 - (R - a)^2}{4ma}\right)\cdot\left(\frac{m^2 + 4b^2 - (R - b)^2}{4mb}\right) = -\frac{3}{4}. \tag{5}$$

Simplifying we obtain

$$\begin{aligned}(R^2-m^2)^2(a+b+c)-(R^2-m^2)\{4R(ab+bc+ca)+3(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)\}\\ + 12R^2abc + 12Rabc(a + b + c) + 9abc(ab + bc + ca) = -\frac{3}{4}\times 16m^2abc.\end{aligned} \tag{6}$$

Multiplying (4) by $4R$ and rearranging, we get

$$12Rabc(a + b + c) = 4R(R^2 - m^2)(ab + bc + ca) - 24R^2abc. \tag{7}$$

Using (7) and the following well-known identity

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 = (a + b + c)(ab + bc + ca) - 3abc,$$

from (6), we obtain

$$\begin{aligned}(R^2-m^2)^2(a+b+c)-4R(R^2-m^2)(ab+bc+ca)-3(R^2-m^2)(a+b+c)(ab+bc+ca)\\ + 9abc(R^2 - m^2) + 12R^2abc + 4R(R^2 - m^2)(ab + bc + ca)\\ - 24R^2abc + 9abc(ab + bc + ca) + 12m^2abc = 0,\end{aligned}$$

which gives

$$\begin{aligned}(R^2 - m^2)^2(a + b + c) - 3(R^2 - m^2)(a + b + c)(ab + bc + ca)\\ - 3abc(R^2 - m^2) + 9abc(ab + bc + ca) = 0,\end{aligned}$$

which can be factorized as

$$[(R^2 - m^2) - 3(ab + bc + ca)][(R^2 - m^2)(a + b + c) - 3abc] = 0.$$

The first factor gives

$$ab + bc + ca = \frac{1}{3}(R^2 - m^2). \tag{8}$$

The second factor gives

$$R^2 - m^2 = \frac{3abc}{a + b + c}, \tag{9}$$

which is not possible, as (4) and (9) give

$$2R(a + b + c) + a^2 + b^2 + c^2 + ab + bc + ca = 0,$$

but the sum of positive terms cannot be zero.

Hence

$$r_1r_3 + r_3r_5 + r_5r_1 = ab + bc + ca = \frac{1}{3}(R^2 - m^2). \tag{10}$$

Similarly, if we join O with O_2, O_4, and O_6, we get $\angle OPO_2 = 60° - \theta$, $\angle OPO_4 = 180° - \theta$, and $\angle OPO_6 = 60° + \theta$. Assuming $60° - \theta = \beta$, we may write $\angle OPO_2 = \beta$, $\angle OPO_4 = 120° + \beta$, and $\angle OPO_6 = 120° - \beta$.
Using cosine law to triangle OPO_2, we get

$$\cos\beta = \frac{m^2 + 4r_2^2 - (R - r_2)^2}{4mr_2}.$$

Using cosine law to triangle OPO_4, we get

$$\cos(120° + \beta) = \frac{m^2 + 4r_4^2 - (R - r_4)^2}{4mr_4}.$$

Using cosine law to triangle OPO_6, we get

$$\cos(120° - \beta) = \frac{m^2 + 4r_6^2 - (R - r_6)^2}{4mr_6}.$$

Hence the two trigonometrical identities mentioned before also hold in this case. Therefore, by similar calculations, we can get

$$r_2r_4 + r_4r_6 + r_6r_2 = \frac{1}{3}(R^2 - m^2). \tag{11}$$

Hence from (10) and (11), we get

$$r_1r_3 + r_3r_5 + r_5r_1 = r_2r_4 + r_4r_6 + r_6r_2 = \frac{1}{3}(R^2 - m^2) = \frac{1}{3}(R^2 - OP^2).$$

□

3. EXPRESSIONS OF R IN TERMS OF r_i

For the above configuration, we present three different expressions for R. Note that (4) holds for pairs $\{r_1, r_3, r_5\}$ and $\{r_2, r_4, r_6\}$. Using (4) and (8), we can write

$$3(ab + bc + ca)^2 = 3abc(2R + a + b + c).$$

It gives

$$R = \frac{(ab + bc + ca)^2 - abc(a + b + c)}{2abc}.$$

Hence in terms of r_1, r_3, and r_5, we can write

$$R = \frac{(r_1r_3 + r_3r_5 + r_5r_1)^2 - r_1r_3r_5(r_1 + r_3 + r_5)}{2r_1r_3r_5}.$$

Similarly, in terms of r_2, r_4, and r_6, we can write

$$R = \frac{(r_2r_4 + r_4r_6 + r_6r_2)^2 - r_2r_4r_6(r_2 + r_4 + r_6)}{2r_2r_4r_6}.$$

Since we have $r_1r_3 + r_3r_5 + r_5r_1 = r_2r_4 + r_4r_6 + r_6r_2$, the above two expressions give

$$R = \frac{r_1r_3r_5(r_1 + r_3 + r_5) - r_2r_4r_6(r_2 + r_4 + r_6)}{2(r_2r_4r_6 - r_1r_3r_5)}.$$

4. An Invariant Relationship

The next relationship we are going to prove holds for all circles either tangent internally or externally. That is why we call it an invariant relationship. Also, the angles between any two successive chords are not necessarily $60°$.

Theorem 4.1. *Through point P inside circle $O(R)$ three straight lines are drawn and they cut the circle $O(R)$ successively, in six points-A_i $(i = 1, 2, \ldots, 6)$. Let six circles $O_i(r_i)$ $(i = 1, 2, \ldots, 6)$ are inscribed into the curvilinear triangles A_iPA_{i+1} for $i = 1, 2, \ldots, 6$, $(A_7 \equiv A_1)$, and six circles $O'_i(\rho_i)$ externally touch $O(R)$ and tangent to rays PA_i and PA_{i+1} for $(i = 1, 2, \ldots, 6)$ where $\angle A_1PA_2 = \alpha_1$, $\angle A_2PA_3 = \alpha_2$ and $\angle A_3PA_4 = \alpha_3$. Then the following relationships*

$$\begin{aligned}&\lambda_1(a_1+a_4)\left(\frac{1}{a_2}+\frac{1}{a_5}\right)+\lambda_2(a_2+a_5)\left(\frac{1}{a_3}+\frac{1}{a_6}\right)+\lambda_3(a_3+a_6)\left(\frac{1}{a_1}+\frac{1}{a_4}\right)\\ =&\lambda_2(a_2+a_5)\left(\frac{1}{a_1}+\frac{1}{a_4}\right)+\lambda_3(a_3+a_6)\left(\frac{1}{a_2}+\frac{1}{a_5}\right)+\lambda_1(a_1+a_4)\left(\frac{1}{a_3}+\frac{1}{a_6}\right)\end{aligned}$$

hold if, all a_i are replaced by r_i or, all a_i are replaced by ρ_i for $i = 1, 2, \ldots, 6$ where $\lambda_j = \cot^2\frac{\alpha_j}{2}$ for $j = 1, 2, 3$.

Proof. **(Internal Case)** Consider an opposite pair of circles $O_1(r_1)$ and $O_4(r_4)$. Let N_1 and N_4 be the orthogonal projections of O_1 and O_4 on A_1A_4, respectively. Join O_1, N_1; O_4, N_4; P, O; P, O_1; P, O_4. Since $\angle O_1PN_1 = \angle O_4PN_4 = \frac{\alpha_1}{2}$, then $PO_1 = kr_1$ and $PO_4 = kr_4$, where $k = \operatorname{cosec}\frac{\alpha_1}{2}$. Note that $OO_1 = R - r_1$ and $OO_4 = R - r_4$.

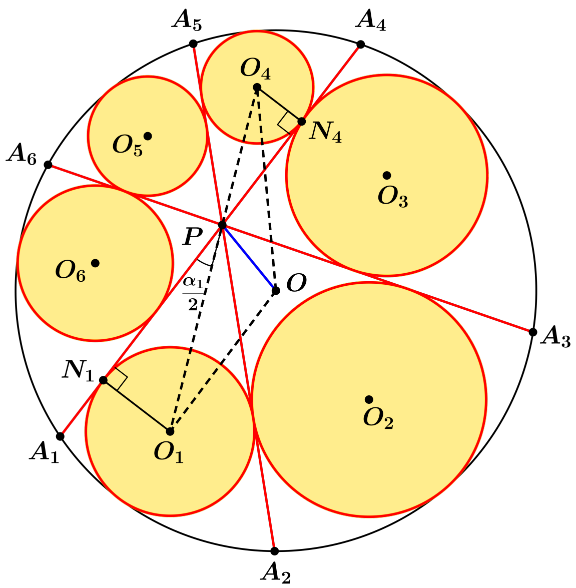

Figure 3.

Applying Stewart's theorem on $\triangle OO_1O_4$ (See Figure 3), we get

$$\begin{aligned}&OO_4^2 \cdot PO_1 + OO_1^2 \cdot PO_4 = O_1O_4(OP^2 + PO_1 \cdot PO_4)\\ \Longrightarrow\ &(R-r_4)^2 \cdot kr_1 + (R-r_1)^2 \cdot kr_4 = k(r_1+r_4)(OP^2 + k^2r_1r_4)\\ \Longrightarrow\ &R^2 - OP^2 = \frac{4Rr_1r_4}{r_1+r_4} + (k^2-1)r_1r_4 = \frac{4Rr_1r_4}{r_1+r_4} + r_1r_4\cot^2\frac{\alpha_1}{2}.\end{aligned}$$

Similarly, considering other opposite pairs $\{O_2(r_2), O_5(r_5)\}$, $\{O_3(r_3), O_6(r_6)\}$ and combining the above result, we may write

$$R^2 - OP^2 = \frac{4Rr_1r_4}{r_1+r_4} + \lambda_1 r_1 r_4 = \frac{4Rr_2r_5}{r_2+r_5} + \lambda_2 r_2 r_5 = \frac{4Rr_3r_6}{r_3+r_6} + \lambda_3 r_3 r_6,$$

where $\lambda_j = \cot^2 \frac{\alpha_j}{2}$ for $j = 1, 2, 3$.

The above expressions can be written as

$$\frac{4R + \lambda_1(r_1+r_4)}{\frac{1}{r_1}+\frac{1}{r_4}} = \frac{4R + \lambda_2(r_2+r_5)}{\frac{1}{r_2}+\frac{1}{r_5}} = \frac{4R + \lambda_3(r_3+r_6)}{\frac{1}{r_3}+\frac{1}{r_6}},$$

which implies

$$\frac{\lambda_1(r_1+r_4) - \lambda_2(r_2+r_5)}{\frac{1}{r_1}+\frac{1}{r_4}-\frac{1}{r_2}-\frac{1}{r_5}} = \frac{\lambda_2(r_2+r_5) - \lambda_3(r_3+r_6)}{\frac{1}{r_2}+\frac{1}{r_5}-\frac{1}{r_3}-\frac{1}{r_6}}.$$

Finally, it gives

$$\lambda_1(r_1+r_4)\left(\frac{1}{r_2}+\frac{1}{r_5}\right) + \lambda_2(r_2+r_5)\left(\frac{1}{r_3}+\frac{1}{r_6}\right) + \lambda_3(r_3+r_6)\left(\frac{1}{r_1}+\frac{1}{r_4}\right)$$
$$=\lambda_2(r_2+r_5)\left(\frac{1}{r_1}+\frac{1}{r_4}\right) + \lambda_3(r_3+r_6)\left(\frac{1}{r_2}+\frac{1}{r_5}\right) + \lambda_1(r_1+r_4)\left(\frac{1}{r_3}+\frac{1}{r_6}\right).$$

External Case: Consider an opposite pair of circles $O_1'(\rho_1)$ and $O_4'(\rho_4)$. Let N_1' and N_4' be the orthogonal projections of O_1' and O_4' on extended A_1A_4, respectively. Join O_1', N_1'; O_4', N_4'; P, O; P, O_1'; P, O_4'. Since $\angle O_1'PN_1' = \angle O_4'PN_4' = \frac{\alpha_1}{2}$, then $PO_1' = k\rho_1$ and $PO_4' = k\rho_4$, where $k = \operatorname{cosec} \frac{\alpha}{2}$. Note that $OO_1' = R + \rho_1$ and $OO_4' = R + \rho_4$.

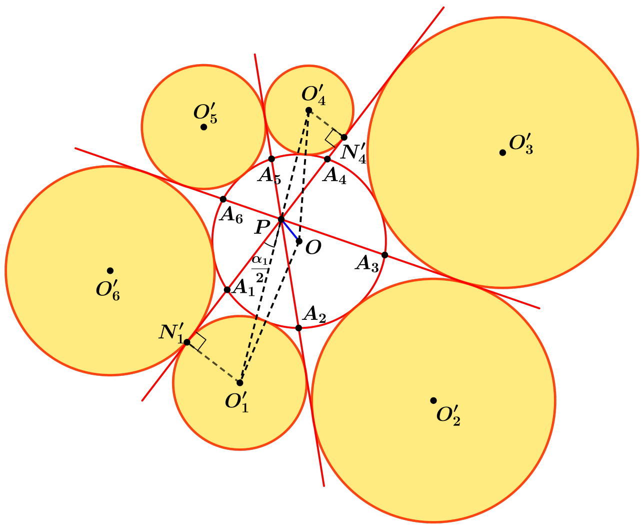

FIGURE 4.

Applying Stewart's theorem on $\triangle OO_1'O_4'$ (See Figure 4), we get

$$OO_4'^2 \cdot PO_1' + OO_1'^2 \cdot PO_4' = O_1'O_4'(OP^2 + PO_1' \cdot PO_4')$$
$$\Longrightarrow (R+\rho_4)^2 \cdot k\rho_1 + (R+\rho_1)^2 \cdot k\rho_4 = k(\rho_1+\rho_4)(OP^2 + k^2\rho_1\rho_4)$$
$$\Longrightarrow R^2 - OP^2 = -\frac{4R\rho_1\rho_4}{\rho_1+\rho_4} + (k^2-1)\rho_1\rho_4 = -\frac{4R\rho_1\rho_4}{\rho_1+\rho_4} + \rho_1\rho_4 \cot^2\frac{\alpha_1}{2}.$$

Hence by exactly similar calculations as in the internal case, we obtain

$$\begin{aligned}&\lambda_1(\rho_1+\rho_4)\left(\frac{1}{\rho_2}+\frac{1}{\rho_5}\right)+\lambda_2(\rho_2+\rho_5)\left(\frac{1}{\rho_3}+\frac{1}{\rho_6}\right)+\lambda_3(\rho_3+\rho_6)\left(\frac{1}{\rho_1}+\frac{1}{\rho_4}\right)\\ =&\lambda_2(\rho_2+\rho_5)\left(\frac{1}{\rho_1}+\frac{1}{\rho_4}\right)+\lambda_3(\rho_3+\rho_6)\left(\frac{1}{\rho_2}+\frac{1}{\rho_5}\right)+\lambda_1(\rho_1+\rho_4)\left(\frac{1}{\rho_3}+\frac{1}{\rho_6}\right).\square\end{aligned}$$

If the angles between any two successive straight lines pass through P are equal then $\alpha_1=\alpha_2=\alpha_3=60°$. So λ_j $(j=1,2,3)$ are also equal. Then can drop λ_j from the above expressions and get λ free expressions [[5], [6]].

5. Generalization

A generalization of **Theorem 4.1** for $2n$ circles is possible by a similar approach. Let's discuss the internal case. The external case is similar.

Theorem 5.1. *Through point P inside circle $O(R)$, n straight lines are drawn and they cut the circle $O(R)$ successively, in $2n$ points-A_i $(i=1,2,\dots,2n)$. Let $2n$ circles $O_i(r_i)$ $(i=1,2,\dots,2n)$ be inscribed into the curvilinear triangles A_iPA_{i+1} for $i=1,2,\dots,2n$ $(A_{2n+1}\equiv A_1)$, and where the angles between two successive straight lines at P are α_p $(p=1,2,\dots,n)$ such that $\angle A_pPA_{p+1}=\alpha_p$. If we choose any three pair of opposite circles $\{O_i(r_i),O_{i+n}(r_{i+n})\}$, $\{O_j(r_j),O_{j+n}(r_{j+n})\}$, and $\{(O_k(r_k),O_{k+n}(r_{k+n})\}$ for $1\le i<j<k\le n$ then the following relationships*

$$\begin{aligned}&\lambda_i(r_i+r_{i+n})\left(\frac{1}{r_j}+\frac{1}{r_{j+n}}\right)+\lambda_j(r_j+r_{j+n})\left(\frac{1}{r_k}+\frac{1}{r_{k+n}}\right)+\lambda_k(r_k+r_{k+n})\left(\frac{1}{r_i}+\frac{1}{r_{i+n}}\right)\\ =&\lambda_j(r_j+r_{j+n})\left(\frac{1}{r_i}+\frac{1}{r_{i+n}}\right)+\lambda_k(r_k+r_{k+n})\left(\frac{1}{r_j}+\frac{1}{r_{j+n}}\right)+\lambda_i(r_i+r_{i+n})\left(\frac{1}{r_k}+\frac{1}{r_{k+n}}\right).\end{aligned}$$

hold where $\lambda_q=\cot^2\frac{\alpha_q}{2}$ $(q=i,j,k)$ are some positive real numbers.

Proof. Consider three pair of opposite circles $\{O_i(r_i),O_{i+n}(r_{i+n})\}$, $\{O_j(r_j),O_{j+n}(r_{j+n})\}$, and $\{(O_k(r_k),O_{k+n}(r_{k+n})\}$ for $1\le i<j<k\le n$.

Applying Stewart's theorem as before we get

$$R^2-OP^2=\frac{4Rr_tr_{t+n}}{r_t+r_{t+n}}+r_tr_{t+n}\cot^2\frac{\alpha_t}{2},\ \ (t=i,j,k).$$

Eliminating R from the above three fractions, we obtain the desired relationship. If we replace all 'r' with 'ρ', i.e., if we consider all external circles then the above theorem also holds and the proof is almost the same. $\square$

6. Relationships between internal and external radii

Now we will explore relationships between internal and external radii. Let's prove the following theorem.

Theorem 6.1. *Through point P inside circle $O(R)$ two straight lines are drawn at an angle α to each other and they cut the circle $O(R)$ successively, in four points-A_i $(i=1,2,3,4)$. If two circles $O_i(r_i)$ $(i=1,2)$ are inscribed into the curvilinear triangles A_jPA_{j+1} $(j=1,2)$, respectively, and two circles $O'_i(\rho_i)$ $(i=1,2)$ externally touch $O(R)$ and tangent to rays PA_j and PA_{j+1} for $j=1,3$, respectively, (See Figure 5) then we have $r_1\rho_2=r_2\rho_1$.*

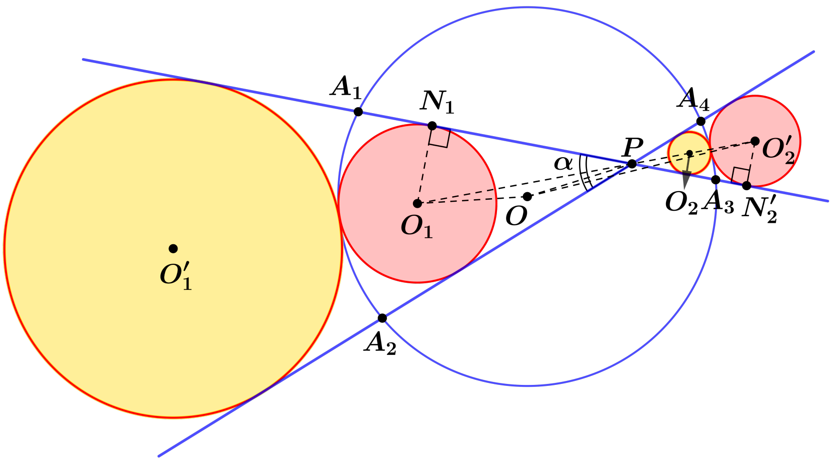

FIGURE 5.

First Proof: Consider the opposite pair of circles $O_1(r_1)$ and $O_2'(\rho_2)$. Let N_1 and N_2' be the orthogonal projections of O_1 and O_2' on extended A_1A_3, respectively. Join O_1, N_1; O_2', N_2'; P, O; P, O_1; P, O_2'. Since $\triangle PO_1N_1 \sim \triangle PO_2'N_2'$, we have $PO_1 : PO_2' = r_1 : \rho_2$. Therefore, we can take $PO_1 = kr_1$ and $PO_2' = k\rho_2$. Note that $OO_1 = R - r_1$ and $OO_2' = R + \rho_2$.

Applying Stewart's theorem on $\triangle OO_1O_2'$ (See Figure 5), we get

$$OO_2'^2 \cdot PO_1 + OO_1^2 \cdot PO_2' = O_1O_2'(OP^2 + PO_1 \cdot PO_2')$$
$$\Longrightarrow (R+\rho_2)^2 \cdot kr_1 + (R-r_1)^2 \cdot k\rho_2 = (kr_1 + k\rho_2)(OP^2 + k^2 r_1\rho_2)$$
$$\Longrightarrow R^2 - OP^2 = (k^2-1)r_1\rho_2.$$

Again, since $\angle A_1PA_2 = \alpha$, we have $\angle A_1PO_1 = \angle N_1PO_1 = \alpha/2$. Therefore, from the right triangle N_1PO_1, we get $PO_1 = r_1 \operatorname{cosec} \frac{\alpha}{2} = kr_1$ and hence $k = \operatorname{cosec} \frac{\alpha}{2}$. Therefore, we have

$$R^2 - OP^2 = r_1\rho_2 \cot^2 \frac{\alpha}{2},$$

which implies

$$r_1\rho_2 = (R^2 - OP^2)\tan^2 \frac{\alpha}{2}.$$

Since $\angle A_3PA_4 = \alpha$, similarly, we obtain

$$r_2\rho_1 = (R^2 - OP^2)\tan^2 \frac{\alpha}{2}.$$

The above two relations imply $\quad r_1\rho_2 = r_2\rho_1.$ □

Second Proof: We can also prove the above Theorem by the Inversion Technique. In this case, we need to choose the circle of center P orthogonal to the circle $O(R)$ as the circle of inversion of the figure. Since point P lies inside the circle $O(R)$, the power of inversion is negative and so the circle of inversion has a radius of imaginary number. According to the theory of inversion, an inversion with center P and negative power $-p$ is equivalent to an inversion with center P and positive power p followed by a reflection in P.

Here $p = \sqrt{PA_1 \cdot PA_3} = \sqrt{PA_2 \cdot PA_4}$. This inversion map circles $O_1(r_1)$ to $O_2'(\rho_2)$ and $O_1'(\rho_1)$ to $O_2(r_2)$. Let N_1, N_2, N_1', and N_2' be the orthogonal projections of O_1, O_2, O_1', and O_2' on A_1A_3, respectively. Join O_1, N_1; O_1', N_1'; O_2, N_2; O_2', N_2'.

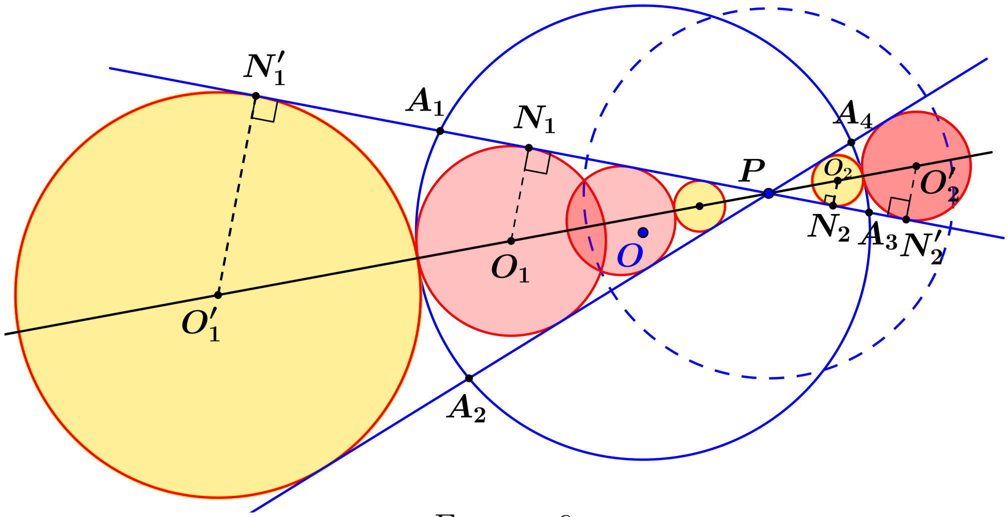

FIGURE 6.

From the theory of inversion, we have (See Figure 6)

$$PN_1 \cdot PN_2' = p^2 = PN_1' \cdot PN_2.$$

Since triangles PO_1N_1, $PO_2'N_2'$, $PO_1'N_1'$, and PO_2N_2 are similar, then from the above relation, it follows that $r_1\rho_2 = r_2\rho_1$. □

Remark. *The above result was also proposed by Koshizuka in 1828 in Tochigi prefecture* [4]. *Instead of internal common tangents of four circles ρ_1, r_1, r_2, and ρ_2, if we take external common tangents (if exist) the same relationship $r_1\rho_2 = r_2\rho_1$ also holds. This version is known as Ohara's Theorem [*[3]*, p.97]. This result can be proved using Stewart's theorem or by inversion as above. See Figure 7.*

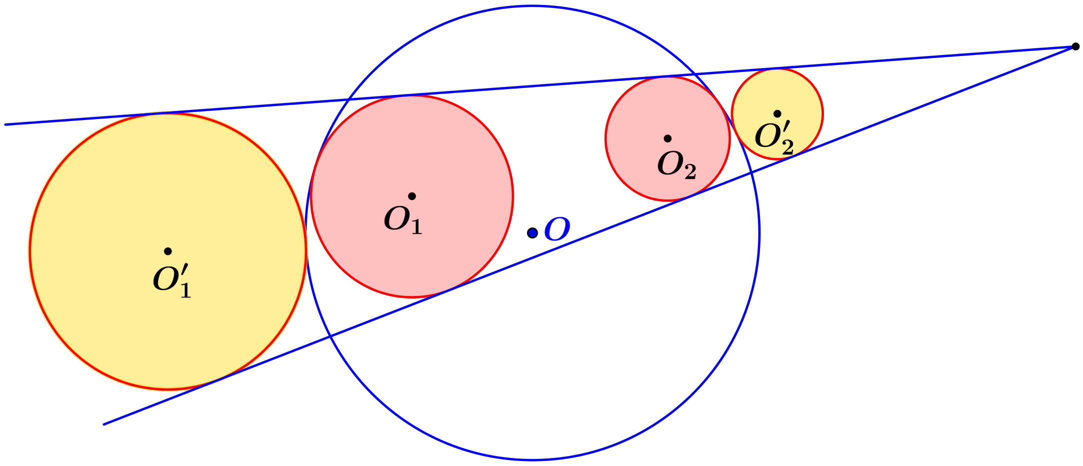

FIGURE 7.

Theorem 6.2. *Through point P inside circle $O(R)$ three straight lines are drawn and they cut the circle $O(R)$ successively, in six points-A_i $(i = 1, 2, \ldots, 6)$. Let six circles $O_i(r_i)$ $(i = 1, 2, \ldots, 6)$ are inscribed into the curvilinear triangles A_iPA_{i+1} for $i = 1, 2, \ldots, 6$, $(A_7 \equiv A_1)$, and six circles $O_i'(\rho_i)$ are externally touch $O(R)$ and tangent to rays PA_i and PA_{i+1} for $(i = 1, 2, \ldots, 6)$ where $\angle A_1PA_2 = \alpha_1$, $\angle A_2PA_3 = \alpha_2$ and $\angle A_3PA_4 = \alpha_3$. Define*

$$E_1 = \lambda_1(r_1 + r_4)\left(\frac{1}{r_2} + \frac{1}{r_5}\right) + \lambda_2(r_2 + r_5)\left(\frac{1}{r_3} + \frac{1}{r_6}\right) + \lambda_3(r_3 + r_6)\left(\frac{1}{r_1} + \frac{1}{r_4}\right),$$

$$E_2 = \lambda_2(r_2+r_5)\left(\frac{1}{r_1}+\frac{1}{r_4}\right)+\lambda_3(r_3+r_6)\left(\frac{1}{r_2}+\frac{1}{r_5}\right)+\lambda_1(r_1+r_4)\left(\frac{1}{r_3}+\frac{1}{r_6}\right),$$

$$E_3 = \lambda_1(\rho_1+\rho_4)\left(\frac{1}{\rho_2}+\frac{1}{\rho_5}\right)+\lambda_2(\rho_2+\rho_5)\left(\frac{1}{\rho_3}+\frac{1}{\rho_6}\right)+\lambda_3(\rho_3+\rho_6)\left(\frac{1}{\rho_1}+\frac{1}{\rho_4}\right),$$

$$E_4 = \lambda_2(\rho_2+\rho_5)\left(\frac{1}{\rho_1}+\frac{1}{\rho_4}\right)+\lambda_3(\rho_3+\rho_6)\left(\frac{1}{\rho_2}+\frac{1}{\rho_5}\right)+\lambda_1(\rho_1+\rho_4)\left(\frac{1}{\rho_3}+\frac{1}{\rho_6}\right).$$

Then we have $E_1 = E_2 = E_3 = E_4$ where $\lambda_j = \cot^2\frac{\alpha_j}{2}$ for $j = 1, 2, 3$.

Proof. From **Theorem 4.1**, we get $E_1 = E_2$ and $E_3 = E_4$. It is enough to prove $E_1 = E_4$. We have $\angle A_1PA_2 = \alpha_1 = \angle A_4PA_5$, $\angle A_2PA_3 = \alpha_2 = \angle A_5PA_6$, and $\angle A_3PA_4 = \alpha_3 = A_6PA_1$. Applying **Theorem 6.1** to quadruples of circles $\{(O_1), (O_4'), (O_4), (O_1')\}$, $\{(O_2), (O_5'), (O_5), (O_2')\}$, and $\{(O_3), (O_6'), (O_6), (O_3')\}$, respectively, we obtain

$$r_1\rho_4 = r_4\rho_1 = (R^2 - OP^2)\tan^2\frac{\alpha_1}{2}, \quad r_2\rho_5 = r_5\rho_2 = (R^2 - OP^2)\tan^2\frac{\alpha_2}{2},$$

and

$$r_3\rho_6 = r_6\rho_3 = (R^2 - OP^2)\tan^2\frac{\alpha_3}{2}.$$

Therefore, we may write

$$(r_1+r_4)\left(\frac{1}{r_2}+\frac{1}{r_5}\right)\cot^2\frac{\alpha_1}{2} = \left(\frac{1}{\rho_4}+\frac{1}{\rho_1}\right)(\rho_5+\rho_2)\cot^2\frac{\alpha_2}{2},$$

$$(r_2+r_5)\left(\frac{1}{r_3}+\frac{1}{r_6}\right)\cot^2\frac{\alpha_2}{2} = \left(\frac{1}{\rho_5}+\frac{1}{\rho_2}\right)(\rho_6+\rho_3)\cot^2\frac{\alpha_3}{2},$$

and

$$(r_3+r_6)\left(\frac{1}{r_4}+\frac{1}{r_1}\right)\cot^2\frac{\alpha_3}{2} = \left(\frac{1}{\rho_6}+\frac{1}{\rho_3}\right)(\rho_1+\rho_4)\cot^2\frac{\alpha_1}{2}.$$

Adding the above three relations, we obtain $E_1 = E_4$, and we are done. □

Corollary 6.1. *For six pair of circles $O_i(r_i)$ and $O_i'(\rho_i)$ for $i = 1, 2, \ldots, 6$, we have*

$$r_1\rho_4 = r_4\rho_1, \quad r_2\rho_5 = r_5\rho_2, \quad \text{and} \quad r_3\rho_6 = r_6\rho_3.$$

Corollary 6.2. *If $\alpha_1 = \alpha_2 = \alpha_3 = 60^\circ$ then we have*

$$r_1\rho_4 = r_4\rho_1 = r_2\rho_5 = r_5\rho_2 = r_3\rho_6 = r_6\rho_3 = (R^2 - OP^2)/3.$$

Theorem 6.3. *Through point P inside circle $O(R)$ three straight lines are drawn at 60° to each other and they cut the circle $O(R)$ successively, into six points-A_i $(i = 1, 2, \ldots, 6)$. If six circles $O_i'(\rho_i)$ $(i = 1, 2, \ldots, 6)$ externally touch $O(R)$ and tangent to rays PA_i and PA_{i+1} for $i = 1, 2, \ldots, 6$, $(A_7 \equiv A_1)$ then we have*

$$\frac{1}{\rho_1\rho_3}+\frac{1}{\rho_3\rho_5}+\frac{1}{\rho_5\rho_1} = \frac{1}{\rho_2\rho_4}+\frac{1}{\rho_4\rho_6}+\frac{1}{\rho_6\rho_2}.$$

Proof. If six circles $O_i(r_i)$ $(i = 1, 2, \ldots, 6)$ are inscribed into the curvilinear triangles A_iPA_{i+1} for $i = 1, 2, \ldots, 6$, $(A_7 \equiv A_1)$ then **Theorem 1.1** gives

$$r_1r_3 + r_3r_5 + r_5r_1 = r_2r_4 + r_4r_6 + r_6r_2.$$

Also, we have

$$r_1\rho_4 = r_4\rho_1 = r_2\rho_5 = r_5\rho_2 = r_3\rho_6 = r_6\rho_3.$$

Eliminating r_i $(i = 1, 2, \ldots, 6)$ from the above two equations, we get

$$\frac{1}{\rho_1\rho_3} + \frac{1}{\rho_3\rho_5} + \frac{1}{\rho_5\rho_1} = \frac{1}{\rho_2\rho_4} + \frac{1}{\rho_4\rho_6} + \frac{1}{\rho_6\rho_2}.$$

□

7. Expressions for R in terms of ρ_i

We find R in terms of ρ_i for $i = 1, 2, \ldots, 6$ when three straight lines make equal angles at P. In this case, **Corollary 6.2** gives

$$r_1\rho_4 = r_4\rho_1 = r_2\rho_5 = r_5\rho_2 = r_3\rho_6 = r_6\rho_3 = \frac{1}{3}(R^2 - OP^2) = k \ \ (say.)$$

From **Theorem 1.1**, we have

$$r_1r_3 + r_3r_5 + r_5r_1 = r_2r_4 + r_4r_6 + r_6r_2 = \frac{1}{3}(R^2 - OP^2) = k.$$

Eliminating r_i from the above two equations we obtain,

$$k = \frac{\rho_2\rho_4\rho_6}{\rho_2 + \rho_4 + \rho_6} = \frac{\rho_1\rho_3\rho_5}{\rho_1 + \rho_3 + \rho_5} = \frac{\rho_2\rho_4\rho_6 - \rho_1\rho_3\rho_5}{(\rho_2 + \rho_4 + \rho_6) - (\rho_1 + \rho_3 + \rho_5)}.$$

Recall that

$$R = \frac{r_1r_3r_5(r_1 + r_3 + r_5) - r_2r_4r_6(r_2 + r_4 + r_6)}{2(r_2r_4r_6 - r_1r_3r_5)}.$$

Changing r_i to ρ_i, we obtain

$$R = \frac{k}{2} \cdot \frac{(\rho_1\rho_3\rho_5)^2(\rho_2\rho_4 + \rho_4\rho_6 + \rho_6\rho_2) - (\rho_2\rho_4\rho_6)^2(\rho_1\rho_3 + \rho_3\rho_5 + \rho_5\rho_1)}{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6(\rho_2\rho_4\rho_6 - \rho_1\rho_3\rho_5)}.$$

Using the value of k, we finally obtain

$$R = \frac{(\rho_1\rho_3\rho_5)^2(\rho_2\rho_4 + \rho_4\rho_6 + \rho_6\rho_2) - (\rho_2\rho_4\rho_6)^2(\rho_1\rho_3 + \rho_3\rho_5 + \rho_5\rho_1)}{2\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6\{(\rho_2 + \rho_4 + \rho_6) - (\rho_1 + \rho_3 + \rho_5)\}}.$$

The configuration stated in **Theorem 1.1** is very rich. It has inspired many people to create their own problems. Several other properties of this configuration can be found on the website [1] of Late Dr. Alexander Bogomolny. He was the man who popularized this problem. So I have dedicated this paper to him.

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