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A Response to an Open Question of Rabinowitz

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Abstract. In [1], Rabinowitz poses an "open question" about the proof of a theorem that makes extensive use of trigonometry. I provide a non-trigonometric proof of the same theorem.

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Mathematics Subject Classification (2020). 51M15, 51M25

1. INTRODUCTION

Rabinowitz [1, pp. 14–15] proves that, in Figure 1, $\frac{1}{r_1}+\frac{1}{w_2}=\frac{1}{r_2}+\frac{1}{w_1}$, where r_i and w_i are the radii of (O_i) and (W_i) for $i=1,2$. He takes $\triangle ABC$, its circumcircle (ABC), and cevian AD as given; AD extended meets the circle a second time in D'; finally, the circles (W_i) are inscribed in the "skewed sectors" BDD' and CDD'.

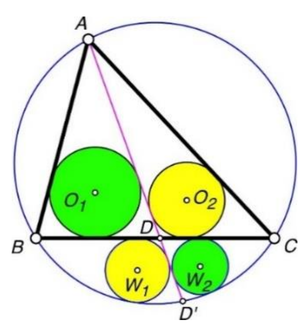

Figure 1. Setting for theorem

Rabinowitz applies the formulae $w = \frac{4R\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\delta}{2}\sin\frac{\epsilon}{2}}{\cos^2\frac{\alpha}{2}}$ and $r = \frac{4R\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sin\frac{\epsilon}{2}\cos\frac{\epsilon}{2}}{\cos\frac{\alpha}{2}}$ for $\triangle ABP$ in Figure 2 to form expressions for w_i and r_i in Figure 1.

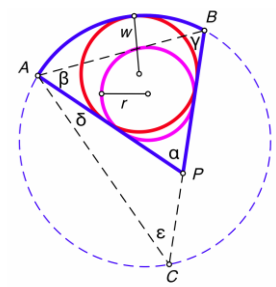

Figure 2. Salient angles in a skewed sector

The angles in the "quarter triangles" (Rabinowitz & Suppa 2022) of cyclic quadrilateral $ABD'C$ correspond to one another as follows:

Quarter triangle	Angles				
$\triangle ABD$	α	β	γ	δ	ε
$\triangle CAD$	$\pi-\alpha$	δ	ε	γ	β
$\triangle D'CD$	α	γ	β	ε	δ
$\triangle BD'D$	$\pi-\alpha$	ε	δ	β	γ

Substituting into $\frac{1}{r_1} + \frac{1}{w_2} = \frac{1}{r_2} + \frac{1}{w_1}$, simplifcation to the extent that the equation becomes obviously true requires first substituting $\delta + \epsilon$ for α and then $\pi - \beta - \gamma - \delta$ for ϵ. Rabinowitz finds this approach unsatisfying and asks whether there is another method of proof "that does not involve a large amount of trigonometric computation requiring computer simplification." The answer is yes, as the following proof shows.

2. A GEOMETRIC SOLUTION

Note first that, denoting the harmonic mean of two numbers as $H(a,b)$, $\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}$ if and only if $H(a,b) = H(c,d)$.

Next, observe that each skewed sector into which the diagonals divide a cyclic quadrilateral is opposite (i.e. non-adjacent to) exactly one other; hence, in Figure 3, there are four ordered pairs incircles (one blue, one black) in opposing quarter triangles and skewed sectors. By transitivity, the theorem is true if only if the harmonic means of the inradii in all four pair are equal.

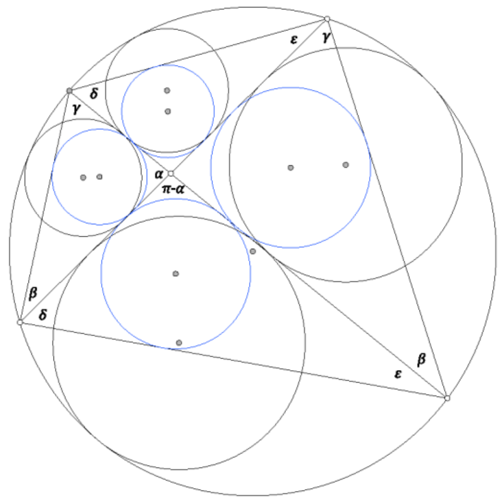

Figure 3. Four skewed sectors in a cyclic quadrilateral

Now, the construction of the common harmonic mean of the inradii in an opposing pair is easy. By reflecting O_1 and W_3, for instance, in one diagonal of the quadrilateral—say V_2V_4— one obtains the primed vertices of an isosceles trapezoid $O_1O_1'W_3W_3'$ (green), of which the chosen diagonal $V_2V_4$is the perpendicular bisector of the parallel sides (bases) O_1O_1' and W_3W_3' (Figure 4). It is easy to prove, by similar triangles, that the line segment H_1H_2 parallel to the trapezoid's bases through the intersection X of its diagonals (also green) is the harmonic mean of the bases. Hence, in the present case, $H_1H_2 = 2H(r_1, w_3)$.

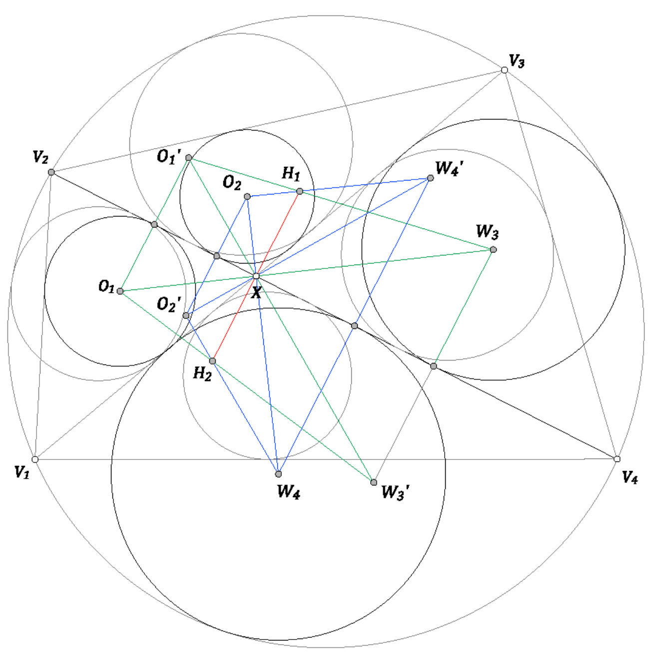

Figure 4. Constructing harmonic means

Note that both V_2V_4 and V_1V_3 also pass through X. V_2V_4 passes through X because it bisects the bases of the trapezoid. Because (O_1) and (W_3) are tangent to both quadrilateral diagonals, V_1V_3 is the reflection of V_2V_4 in O_1W_3, so it too passes through X.

Now select another pair of opposing incenters, e.g. O_2 and W_4. O_2W_4 passes through X because (O_2) and (W_4) are tangent to both quadrilateral diagonals. Denote the points where O_2H_1 and W_4H_2 meet the reflection of O_2W_4 in V_2V_4 as W_4' and O_2', respectively. $O_2O_2'W_4W_4'$ (blue) is a second isosceles trapezoid because $O_2W_4O_2' \cong O_2W_4'O_2'$. Its bases are parallel to the bases of the first, and have lengths $2r_2$ and $2w_4$; its diagonals (also blue) concur in X; therefore $H_1H_2 = 2H(r_2, w_4)$, which proves the theorem.

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