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A Note on a Chain of Contact Circles inside a Circular Segment

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Abstract. Consider a chain of n contact circles inside a circular segment. We will give some interesting invariant relationships among the radii of contact circles and will derive an explicit formula for the radius of the n-th circle in the chain in terms of the radii of the first three contact circles and discuss a special case.

Keywords. Chain of contact circles, Wasan Geometry.

Mathematics Subject Classification (2010). 51M04.

1. Introduction

Consider a chain of contact circles of radii r_i, $i = 1, 2, \cdots n$, on a chord of a circle of radius R and are inscribed in the circular segment as shown in Figure 1. We will prove some invariant relationships among r_i and derive the expression for r_n as a function of r_1, r_2, and r_3. We will also investigate a special case when the chord becomes a diameter of the circle.

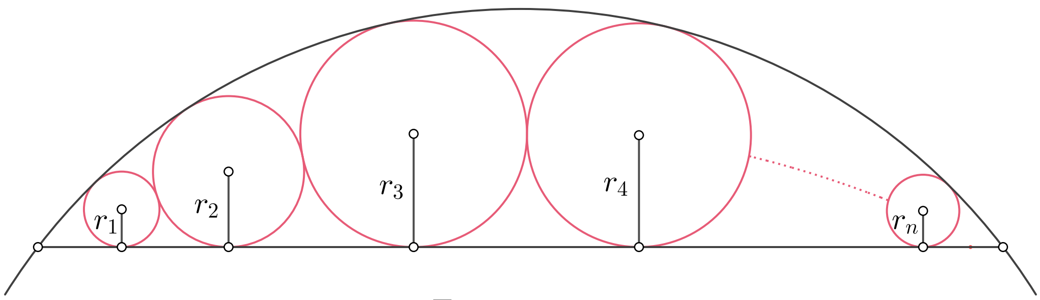

Figure 1

We will use the following result from Wasan geometry [1] (See Figure 2).

Assume that circle C with radius R is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t. If two externally touching *circles C_1 and C_2 with radii r_1 and r_2 also touch the chord t and the other arc of the circle C internally, then h, R, r_1, and r_2 are related by*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2R}{r_1 r_2 h}}. \tag{1}$$

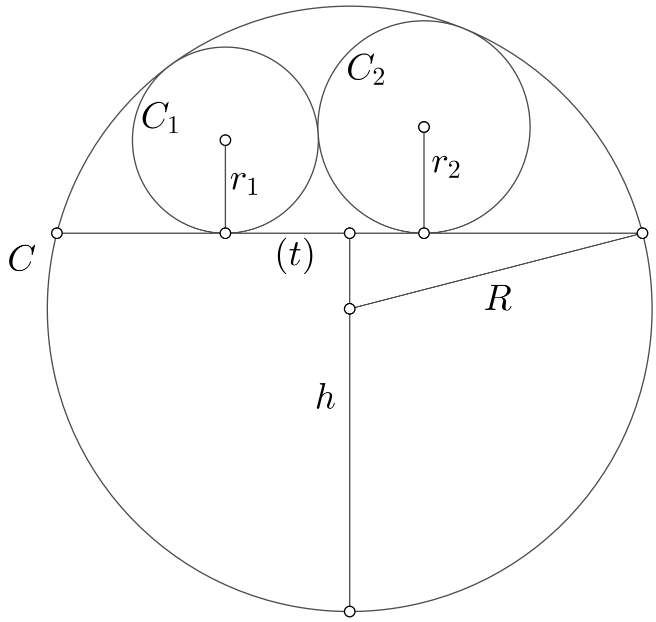

Figure 2

2. Some invariant relationships

The relationship (1) is valid for any two successive contact circles in the chain. So for radii r_2 and r_3, we have

$$\frac{1}{r_2} + \frac{1}{r_3} + \frac{2}{h} = 2\sqrt{\frac{2R}{r_2 r_3 h}}. \tag{2}$$

Subtracting (2) from (1), and simplifying we get

$$\sqrt{r_2}\left(\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_3}}\right) = 2\sqrt{\frac{2R}{h}} = \text{constant}. \tag{3}$$

Relation (3) suggests that for r_{n-1}, r_n, and r_{n+1}, the following expression

$$\sqrt{r_n}\left(\frac{1}{\sqrt{r_{n-1}}} + \frac{1}{\sqrt{r_{n+1}}}\right), \tag{4}$$

is constant for any positive integer $n \geq 2$.

Moreover, the relation (4) holds for other chains, i.e., if s_{n-1}, s_n, and s_{n+1} are three successive radii in another chain in the same segment, then the following relation (See Figure 3),

$$\sqrt{r_n}\left(\frac{1}{\sqrt{r_{n-1}}} + \frac{1}{\sqrt{r_{n+1}}}\right) = \sqrt{s_n}\left(\frac{1}{\sqrt{s_{n-1}}} + \frac{1}{\sqrt{s_{n+1}}}\right),$$

holds as the right-hand side of the above relation gives the same constant $2\sqrt{\frac{2R}{h}}$.

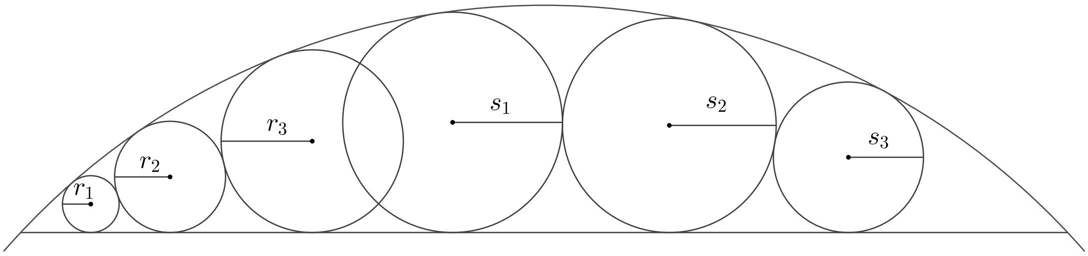

FIGURE 3

By virtue of (4), we can write,

$$\sqrt{r_2}\left(\frac{1}{\sqrt{r_1}}+\frac{1}{\sqrt{r_3}}\right)=\sqrt{r_3}\left(\frac{1}{\sqrt{r_2}}+\frac{1}{\sqrt{r_4}}\right), \tag{5}$$

which can be written as

$$\frac{1}{\sqrt{r_1 r_3}}-\frac{1}{r_2}=\frac{1}{\sqrt{r_2 r_4}}-\frac{1}{r_3}.$$

It suggests that for r_{n-1}, r_n, and r_{n+1} the expression

$$\frac{1}{\sqrt{r_{n-1}r_{n+1}}}-\frac{1}{r_n},$$

is constant and we can also determine the value of this constant.
Dividing (1) by (2), we get,

$$\sqrt{r_1}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{2}{h}\right)=\sqrt{r_3}\left(\frac{1}{r_2}+\frac{1}{r_3}+\frac{2}{h}\right),$$

which gives

$$\frac{2}{h}=\frac{1}{\sqrt{r_2 r_3}}-\frac{1}{r_2}. \tag{6}$$

Hence

$$\frac{1}{\sqrt{r_{n-1}r_{n+1}}}-\frac{1}{r_n}=\frac{2}{h}.$$

Using (3) and (6), we can easily deduce

$$R=\frac{r_2^2(\sqrt{r_1}+\sqrt{r_3})^2}{4\sqrt{r_1 r_3}(r_2-\sqrt{r_1 r_3})}.$$

The above relation also holds for any three successive radii r_{n-1}, r_n, and r_{n+1}.

3. r_n IN TERMS OF r_1, r_2, r_3.

From relation (5), solving for r_4, we get

$$r_4=\frac{r_1 r_2 r_3^2}{\left(r_2\sqrt{r_1}+r_2\sqrt{r_3}-r_3\sqrt{r_1}\right)^2}.$$

This relation holds for four successive radii in the chain but it does not give r_n in terms of r_1, r_2, r_3. To achieve this we proceed as follows.

Assume the constant $\sqrt{r_2}\left(\frac{1}{\sqrt{r_3}}+\frac{1}{\sqrt{r_1}}\right)=\lambda$ and $d_n=\frac{1}{\sqrt{r_n}}$.

By virtue of (4) and for radii r_{n-2}, r_{n-1}, and r_n we can write

$$d_n + d_{n-2} = \lambda d_{n-1}.$$

So, the characteristic equation is

$$t^2 - \lambda t + 1 = 0.$$

If α, β are the roots of this equation, then we have

$$d_n = A_1\alpha^n + A_2\beta^n,$$

where

$$\alpha = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}, \qquad \beta = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}, \tag{7}$$

A_1 and A_2 are constants to be determined.
For $n = 1$, and $n = 2$ we have,

$$d_1 = A_1\alpha + A_2\beta, \qquad d_2 = A_1\alpha^2 + A_2\beta^2.$$

By solving the above two equations, we obtain

$$A_1 = \frac{d_2 - \beta d_1}{\alpha(\alpha - \beta)}, \quad A_2 = \frac{d_2 - \alpha d_1}{\beta(\beta - \alpha)}.$$

Therefore,

$$d_n = \frac{(d_2 - \beta d_1)}{(\alpha - \beta)}\alpha^{n-1} - \frac{(d_2 - \alpha d_1)}{(\alpha - \beta)}\beta^{n-1}.$$

Since $\alpha\beta = 1$, on simplifying the above expression, we obtain

$$d_n = \frac{1}{\alpha - \beta}\left\{d_2(\alpha^{n-1} - \beta^{n-1}) - d_1(\alpha^{n-2} - \beta^{n-2})\right\}.$$

Changing $d_k = \dfrac{1}{\sqrt{r_k}}$, for $k = 1, 2, n$ and solving for r_n, we obtain

$$r_n = \frac{(\alpha - \beta)^2 r_1 r_2}{\left\{\sqrt{r_1}(\alpha^{n-1} - \beta^{n-1}) - \sqrt{r_2}(\alpha^{n-2} - \beta^{n-2})\right\}^2}, \quad (n \geq 4), \tag{8}$$

where α and β can be found from (7) as

$$\alpha, \beta = \frac{\sqrt{r_2}(\sqrt{r_1} + \sqrt{r_3}) \pm \sqrt{r_2(\sqrt{r_1} + \sqrt{r_3})^2 - 4r_1r_3}}{2\sqrt{r_1r_3}}.$$

Using above expressions for α, β in (8), we can ontain $r_n = f(r_1, r_2, r_3)$.

4. A special case

We will investigate a special case when the chord becomes a diameter of the circle of radius R. Note that in this case $h = R$.
Substituting $h = R$ in (3), we get

$$\sqrt{r_2}\left(\frac{1}{\sqrt{r_3}} + \frac{1}{\sqrt{r_1}}\right) = 2\sqrt{2}. \tag{9}$$

Using (9) and for three contact circles r_{n-2}, r_{n-1}, and r_n, we can write

$$\frac{1}{\sqrt{r_n}} + \frac{1}{\sqrt{r_{n-2}}} = \frac{2\sqrt{2}}{\sqrt{r_{n-1}}} \qquad \text{or,} \qquad d_n + d_{n-2} = 2\sqrt{2}d_n,$$

using previous notations. Clearly in this case, $\lambda = 2\sqrt{2}$.
Following the same procedure as before, in this case, we obtain

$$r_n = \frac{4r_1r_2}{\left\{\sqrt{r_1}(\alpha^{n-1} - \beta^{n-1}) - \sqrt{r_2}(\alpha^{n-2} - \beta^{n-2})\right\}^2}, \quad (n \geq 3),$$

where

$$\alpha = \sqrt{2} + 1, \quad \text{and} \quad \beta = \sqrt{2} - 1.$$

Clearly, in this case, we can express r_n as a function of r_1 and r_2.
Also, here we can express R as a function of any two successive radii r_n, r_{n+1} as

$$R = \frac{2r_nr_{n+1}}{2\sqrt{2r_nr_{n+1}} - (r_n + r_{n+1})}.$$

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[2] https://www.facebook.com/103907057666827/photos/a.103973994326800/250801726310692/