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The arbelos in Wasan geometry: Saitoh's problem

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Abstract. We generalize a problem in Wasan geometry involving an arbelos proposed by Saitoh in 1811.

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1. Introduction

For a point C on the segment AB such that $|BC| = 2a$, $|CA| = 2b$ and $|AB| = 2c$, we consider an arbelos formed by the three semicircles α, β and γ of diameters BC, CA and AB, respectively, constructed on the same side of AB (see Figure 1). Let t be the tangent of α from the point A. In this article we generalize the following problem proposed by Saitoh (斎藤清馨) in 1811 [2] (see Figure 2).

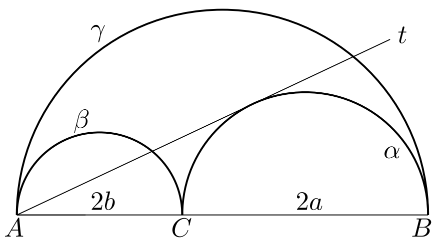

Figure 1.

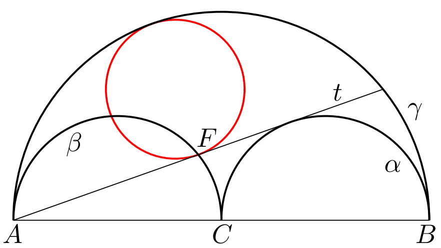

Figure 2.

Problem 1. Assume that $a = b$ and t meets β again in a point F. Show that the radius of the circle touching t at F and γ internally equals $c/3$.

The problem also shows $|CF| = c/3$. The arbelos in Wasan geometry is usually indicated by three circles so that the line joining the three centers of the circles are vertical. But the figure of the problem in [2] is described by three semicircles with the horizontal line passing through their centers just as shown in Figure 2.

2. GENERALIZATION

We generalize Problem 1. We use a rectangular coordinate system with origin C such that the farthest point on α from AB has coordinates (a, a). Assume that Z is a point on the line AB and F is the foot of perpendicular from Z to t. The mirror images of α and γ in AB are denoted by α' and γ', respectively. We consider the two circles δ_z and ε_z of radii d_z and e_z, respectively, such that they touch t at F and one touches γ and the other touches γ' and $e_z \leq d_z$ (see Figure 3). The two circles are said to be *determined by* Z. Notice $c = a + b$. The circle $\gamma \cup \gamma'$ has an equation

$$\gamma(x, y) = (x - 2a)(x + 2b) + y^2 = 0.$$

Let $m = a/(2\sqrt{bc})$. The line t has an equation

$$t(x, y) = (x + 2b)m - y = 0. \tag{1}$$

And the line ZF has an equation

$$z_f(x, y) = (x - z) + my = 0.$$

Assume that Z and the center of δ_z have coordinates $(z, 0)$ and (x_d, y_d).

Theorem 1. *The following relations holds.*

$$d_z = \frac{|\gamma(z, 0)|}{b + c}, \quad e_z = \frac{b}{c} d_z. \tag{2}$$

Proof. Assume that Z lies between A and B. Then δ_z and ε_z touch γ' and γ internally, respectively (see Figure 3). Assume that the perpendicular from the center of δ_z to AB meets t in a point of coordinates (x_d, y'). Then $y_d = y' - k$ for a positive real number k. Then $t(x_d, y_d) = t(x_d, y') + k = k > 0$ by (1). Therefore we have $t(x_d, y_d)/\sqrt{m^2 + 1} = d_z$. Also we have $(x_d - (a - b))^2 + y_d^2 = (c - d_z)^2$ and $z_f(x_d, y_d) = 0$. Eliminating x_d, y_d from the three equations, and solving the resulting equation for d_z, we get

$$d_z = -\frac{(z - 2a)(z + 2b)}{b + c} = \frac{|\gamma(z, 0)|}{b + c}.$$

Similarly we get $e_z = (b/c)d_z$. Hence we get (2). The rest of the theorem is proved similarly. $\square$

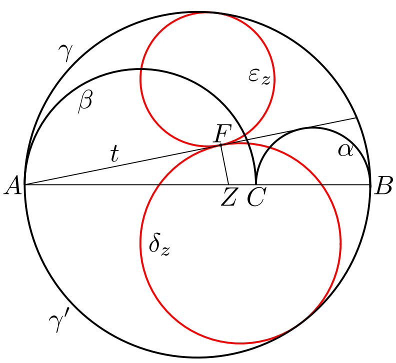

Figure 3.

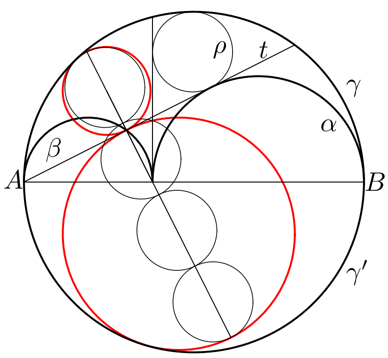

Figure 4: The case $Z = C$.

Corollary 1. $d_z + e_z = \dfrac{|\gamma(z, 0)|}{c}$.

Let ρ be the circle touching α externally and γ internally and the radical axis of α and β from the side opposite to A. The circle ρ is well-known as one of the twin circles of Archimedes, and has radius ab/c [1]. We consider the case $Z = C$. Then F lies on β as in Problem 1. In this case we get $d_z + e_z = 4ab/c$ by the corollary, which equals four times the radius of ρ (see Figure 4). Theorem 1 shows that the ratio of the radii of δ_z and ε_z are constant if $Z \neq A$ and $Z \neq B$.

3. Axis

In this section we consider one of the external common tangents of the circles determined by the point Z which is perpendicular to AB. Let x_e be the x-coordinate of the center of the circle ε_z.

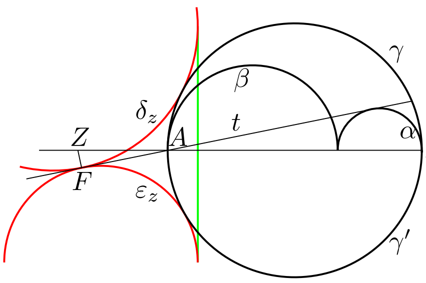

Figure 5: $z \leq -2b$.

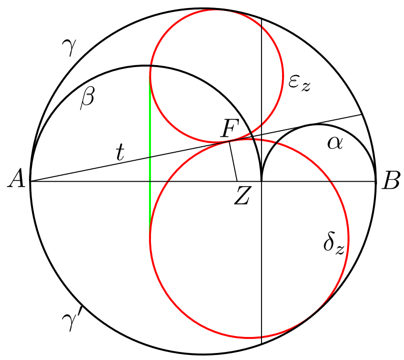

Figure 6: $-2b < z < 2a$.

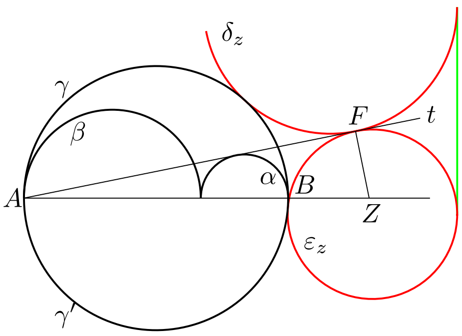

Figure 7: $2a \leq z$.

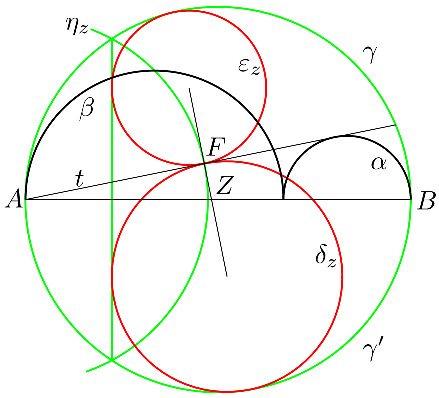

Figure 8.

Theorem 2. *The following statements hold.*
(i) *One of the external common tangent of the circles determined by Z is perpendicular to AB, which is represented by the equation*

$$x = h_z = \frac{2b(z-a)(z+a+4b)}{(b+c)^2}.$$

(ii) *If Z lies between A and B, then $x_d - d_z = x_e - e_z = h_z$, otherwise $x_d + d_z = x_e + e_z = h_z$.*

Proof. Assume $z \leq -2b$ or $2a \leq z$ (see Figures 5 and 7). Since δ_z touches t from the side opposite to B, we get $t(x_d, y_d) \leq 0$. Hence we have

$$\frac{t(x_d, y_d)}{\sqrt{m^2+1}} = -d_z, \quad z_f(x_d, y_d) = 0.$$

Eliminating y_d from the two equations, we have

$$x_d = \frac{4b^2z + a(2b-z)z + 2a^2(b+z)}{(b+c)^2}.$$

Similarly we have

$$x_e = \frac{b(4b^2z + az(z+10b) + 2a^2(z-3b) - 2a^3)}{c(b+c)^2}.$$

We have the same results in the case $-2b < z < 2a$ similarly. Then we get

$$x_d + \frac{\gamma(z,0)}{b+c} = x_e + \frac{b}{c}\frac{\gamma(z,0)}{b+c} = h_z.$$

Hence we have $x_d - d_z = x_e - e_z = h_z$ if $-2b < z < 2a$ (see Figure 6), and $x_d + d_z = x_e + e_z = h_z$ if $z < -2b$ or $2a < z$ by (2). The proof is complete. □

The common tangent of the circles determined by the point Z perpendicular to AB is called the *axis* of Z. If $Z = A$ or $Z = B$, then the circles determined by Z degenerate to the point A or B. In this case we consider that the axis of Z is the perpendicular to AB passing through A or B, respectively. Let η_z be the circle of center A passing through F (see Figure 8).

Theorem 3. *The axis of Z coincides with the radical axis of the circles γ and η_z.*

Proof. Let (x_f, y_f) be the coordinates of F. Solving the equations $t(x_f, y_f) = 0$ and $z_f(x_f, y_f) = 0$, we have

$$(x_f, y_f) = \left(\frac{-2a^2b + 4bcz}{(b+c)^2}, \frac{2a\sqrt{bc}(2b+z)}{(b+c)^2}\right). \tag{3}$$

While the circle ζ_z is represented by the equation

$$\zeta_z(x,y) = (x+2b)^2 + y^2 - (x_f+2b)^2 - y_f^2.$$

This implies $\zeta_z(x,y) - \gamma(x,y) = 2c(x - h_z)$. □

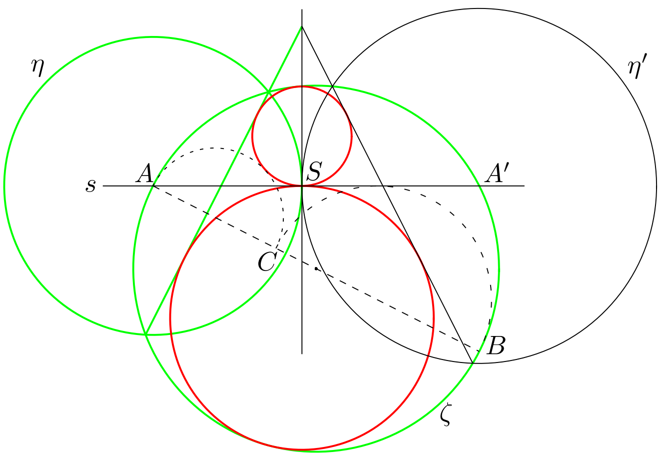

Figure 9.

The next theorem shows that Theorem 3 is not a theorem for the arbelos (see Figure 9).

Theorem 4. *For a point S on a secant s of a circle ζ, let η be a circle of center at one of the points of intersection of ζ and s passing through S. Then the radical axis of ζ and η coincides with one of the external common tangents of the two circles touching s at S and ζ.*

Proof. Assume that s meets ζ again in a point A' and A is the center of η and AB is a diameter of ζ. Let $t = s$, $F = S$ and $\eta_z = \eta$. If γ is the semicircle of diameter AB containing A' and Z is the point of intersection of AB and the perpendicular to s at S, then we can construct Figure 8 referred in Theorem 3 with this figure. □

Let η' be the circle of center A' passing through S in the proof. Then the two external common tangents of the two circles touching s at S and ζ meets in the radical center of the circles ζ, η and η'. Therefore the perpendicular to s at S passes through this point (see Figure 9).

4. Common axis

From now on we assume that Z_1 and Z_2 are two distinct points on the line AB having x-coordinates z_1 and z_2, respectively. In this section we consider the case in which the two points share a common axis. Let F_i be the foot of perpendicular from Z_i to t. The next theorem gives a condition under which the points Z_1 and Z_2 have a common axis (see Figures 10 and 11).

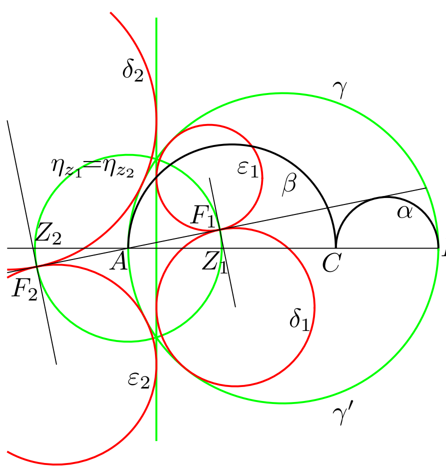

Figure 10.

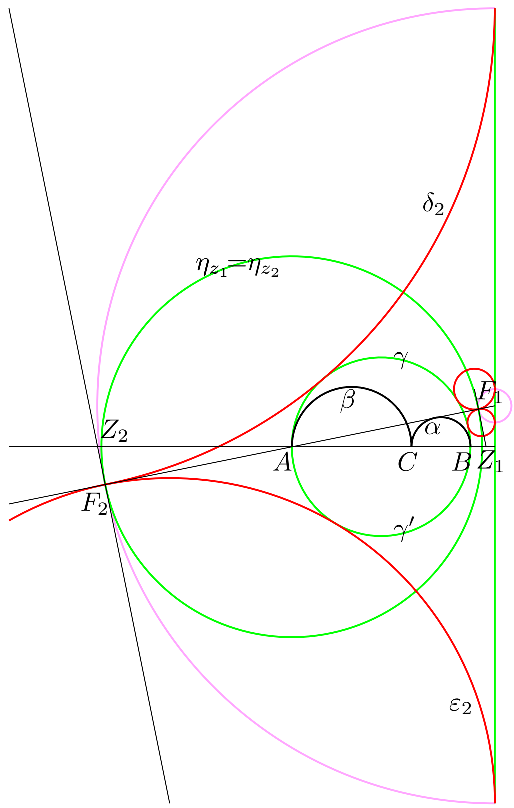

Figure 11.

Theorem 5. *Two distinct points Z_1 and Z_2 have a common axis if and only if the point A coincides with the midpoint of Z_1Z_2.*

Proof. By Theorem 2, Z_1 and Z_2 have a common axis if and only if $(z_1 - a)(z_1 + a + 4b) = (z_2 - a)(z_2 + a + 4b)$, which is equivalent to $(z_1 - z_2)(z_1 + z_2 + 4b) = 0$. □

Assume that the point A coincides with the midpoint of Z_1Z_2. The circle of center at the point of intersection of t and the common axis of Z_1 and Z_2 and passing through the point F_i is orthogonal to the two circles determined by Z_i, which are not indicated in Figures 10 but in Figures 11 in pink. Notice that the right triangles AZ_1F_1 and AZ_2F_2 are congruent. In this case the circles η_{z_1} and η_{z_2} coincide, i.e., the circle $\eta_{z_1} = \eta_{z_2}$ is the circle of diameter F_1F_2, which is orthogonal to the circles determined by Z_1 and Z_2. The common axis is the radical axis of γ and this circle by Theorem 3.

In this case, if exactly one of Z_1 and Z_2 lies between A and B, then one of F_1 and F_2 lies insides of γ, and the common axis intersects γ and the circles determined by Z_1 touch the axis from the side opposite to the circles determined by Z_2 by Theorem 2(ii). Therefore the circles determined by one of Z_1 and Z_2 touch γ externally and the other two circles touch γ internally (see Figure 10). If both Z_1 and Z_2 do not lie between A and B, then the common axis does not intersect γ and the circles determined by Z_1 and Z_2 touch the axis from the same side, and they touch γ externally (see Figure 11). The next theorem is rather obvious.

Theorem 6. *There are two distinct points Z_1 and Z_2 having a common axis represented by the equation $x = h$ if and only if $-2b < h$.*

Proof. By Theorem 2, $h = h_z$ is equivalent to

$$2bz^2 + 8b^2z - (2ab(a + 4b) + (a + 2b)^2h) = 0. \tag{4}$$

We consider (4) as a quadratic equation with unknown z. Then there are two distinct points Z_1 and Z_2 having the common axis represented by $x = h$ if and only if (4) has two distinct real solutions. While the discriminant of (4) equals $8b(a + 2b)^2(2b + h)$. The proof is complete. □

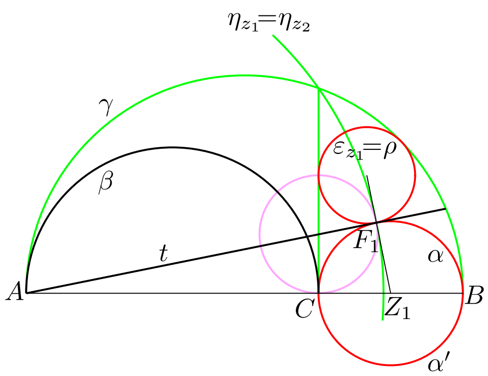

Figure 12: $z_1 = a$.

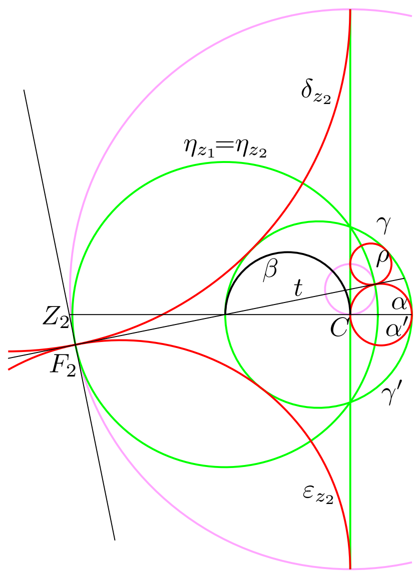

Figure 13: $z_2 = -a - 4b$.

We now consider the special case in which the common axis coincides with the radical axis of α and β (see Figures 12 and 13). This case happens if and only if $\{z_1, z_2\} = \{a, -a-4b\}$ by Theorem 2. Let $z_1 = a$ and $z_2 = -a-4b$. Then the point Z_1 coincides with the center of α. The circle δ_{z_1} coincides with the circle $\alpha \cup \alpha'$. The circle ε_{z_1} coincides with the Archimedean circle ρ defined in Section 2, for F_1 is the point of tangency of α and ρ and t [1]. The circle of center at the point of intersection of t and the common axis and passing through the point F_1 is orthogonal to β and touches AB at C.

5. Common axis with a pair of congruent circles

There arises a problem of determining the case in which Z_1 and Z_2 have a common axis and the two circles determined by Z_1 are congruent to the two circles determined by Z_2. However the next theorem shows that there is no such case.

Theorem 7. *For distinct two points Z_1 and Z_2, there is no case such that they share a common axis and the two circles determined by Z_1 are congruent to the two circles determined by Z_2.*

Proof. Assume that the two points share an axis and the circles determined by Z_1 are congruent to the circles determined by Z_2. If both Z_1 and Z_2 do not lie between A and B, then the circles determined by Z_1 are not congruent to the circles determined by Z_2 (see Figure 11). Hence exactly one of Z_1 and Z_2 lies between A and B by Theorem 2(ii). This implies that $\gamma(z_1, 0)$ and $\gamma(z_2, 0)$ have different signs. Therefore we have $\gamma(z_1, 0) + \gamma(z_2, 0) = 0$ and $z_1 + z_2 = -4b$ by Theorems 1 and 5. Solving the two equations, we get $z_1 = z_2 = -2b$, a contradiction. □

We now consider the case in which the points Z_1 and Z_2 have a common axis and one of the circles determined by Z_1 is congruent to one of the circles determined by Z_2 by the theorem (see Figures 14 and 15, where the congruent circles are described in blue and the captions will be explained later).

Theorem 8. *Two points Z_1 and Z_2 ($z_1 > z_2$) share a common axis and one of the circles determined by Z_1 is congruent to one of the circles determined by Z_2 if and only if they satisfy the following condition:*
(i) $z_1 = -2b + \dfrac{2ca}{b+c}$ *and* $z_2 = -2b - \dfrac{2ca}{b+c}$.
or
(ii) $z_1 = -2b + \dfrac{2c(b+c)}{a}$ *and* $z_2 = -2b - \dfrac{2c(b+c)}{a}$.
In this event the circles δ_{z_1} and ε_{z_2} are congruent and have the same radius

$$\frac{8abc^2}{(b+c)^3} \ \textit{if (i) holds and} \ \frac{8bc^2}{a^2} \ \textit{if (ii) holds.}$$

Proof. Assume that Z_1 and Z_2 share a common axis. Then $z_1 = -2b + z$ and $z_2 = -2b - z$ for a positive real number z by Theorem 5. Theorem 1 shows that if the circles δ_{z_1} and δ_{z_2} are congruent then the circles ε_{z_1} and ε_{z_2} are also congruent, and conversely. But this case never happens by Theorem 7. Hence it is sufficient to consider the case where the circles δ_{z_1} and ε_{z_2} or the circles ε_{z_1} and δ_{z_2} are congruent. The circles δ_{z_1} and ε_{z_2} are congruent if and only if

$$|\gamma(-2b+z, 0)| = b|\gamma(-2b-z, 0)|/c$$

by Theorem 1. This is equivalent to

$$z(c|z-2c|-b|z+2c|)=0.$$

The last equation has two positive solutions $z=2ca/(b+c)$ and $z=2c(b+c)/a$. If ε_{z_1} and δ_{z_2} are congruent, we have

$$b|\gamma(-2b+z,0)|/c=|\gamma(-2b-z,0)|,$$

which is equivalent to

$$z(b|z-2c|-c|z+2c|)=0.$$

However the last equation for z has no positive solution. Therefore ε_{z_1} and δ_{z_2} are not congruent in any case. The rest of the theorem follows by (2). □

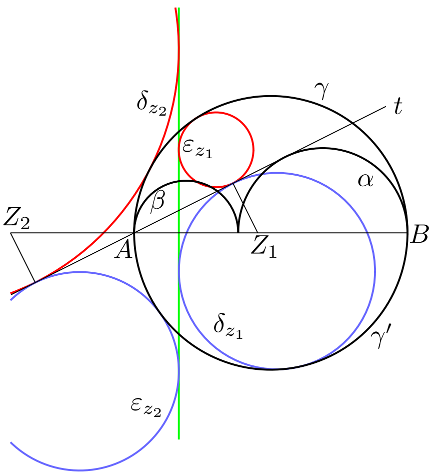

Figure 14: $z_1=-2b+\dfrac{2ca}{b+c}$.

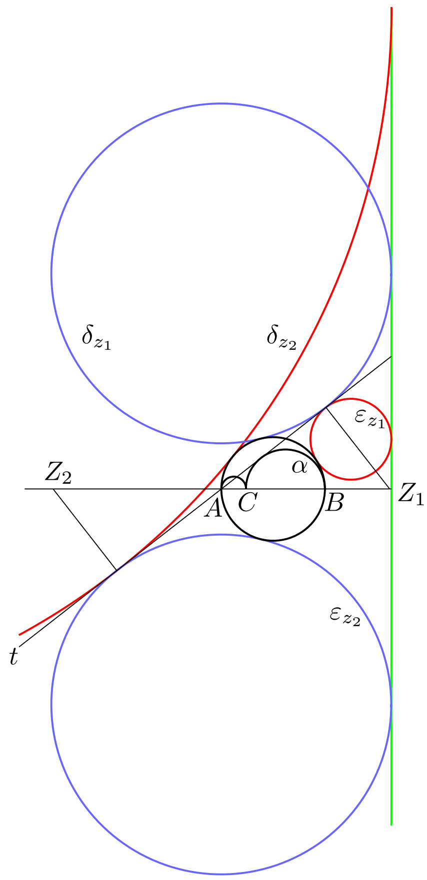

Figure 15: $z_1=-2b+\dfrac{2c(b+c)}{a}$.

If $z=2ca/(b+c)$, then $2a-z_1=2a-(-2b+z)=4bc/(b+c)>0$, which implies $-2b<z_1<2a$. If $z=2c(b+c)/a$, then $z_1-2a=-2b+z-2a=4bc/a>0$, which implies $2a<z_1$. Therefore Figures 14 and 15 denote the cases (i) and (ii), respectively.

References

[1] C. W. Dodge, T. Schoch, P. Y. Woo, P. Yiu, Those ubiquitous Archimedean circles, Math. Mag., **72** (1999) 202–213.

[2] Shimura (志村昌義) et al. ed., Kiōshū (淇澳集), Tohoku University Digital Collection.