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Relationships Between Six Excircles

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Abstract. If P is a point inside $\triangle ABC$, then the cevians through P divide $\triangle ABC$ into smaller triangles of various sizes. We give theorems about the relationship between the radii of certain excircles of some of these triangles.

Keywords. Euclidean geometry, triangle geometry, excircles, exradii, cevians.

Mathematics Subject Classification (2010). 51M04.

1. Introduction

Let P be any point inside a triangle ABC. The cevians through P divide $\triangle ABC$ into six smaller triangles. In a previous paper [5], we found relationships between the radii of the circles inscribed in these triangles.

For example, if P is at the orthocenter H, as shown in Figure 1, then we found that $r_1r_3r_5 = r_2r_4r_6$, where the r_i are radii of the incircles as shown in the figure.

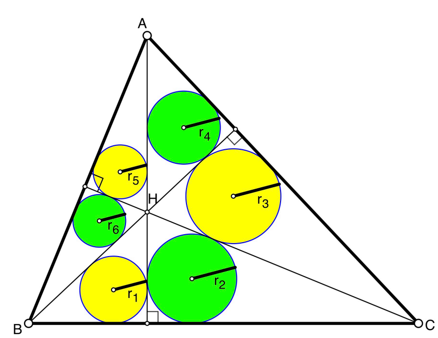

Figure 1. $r_1r_3r_5 = r_2r_4r_6$

In this paper, we will find similar results using excircles instead of incircles. When the cevians through a point P interior to a triangle ABC are drawn, many smaller triangles of various sizes are formed. These triangles have three excircles each. In this paper, we only investigate two configurations of six excircles. These two configurations are shown in Figure 2. Note that in configuration 1, the circle with radius r_1 is an excircle of $\triangle BAD$. In configuration 2, the circle with radius r_1 is an excircle of $\triangle BPD$.

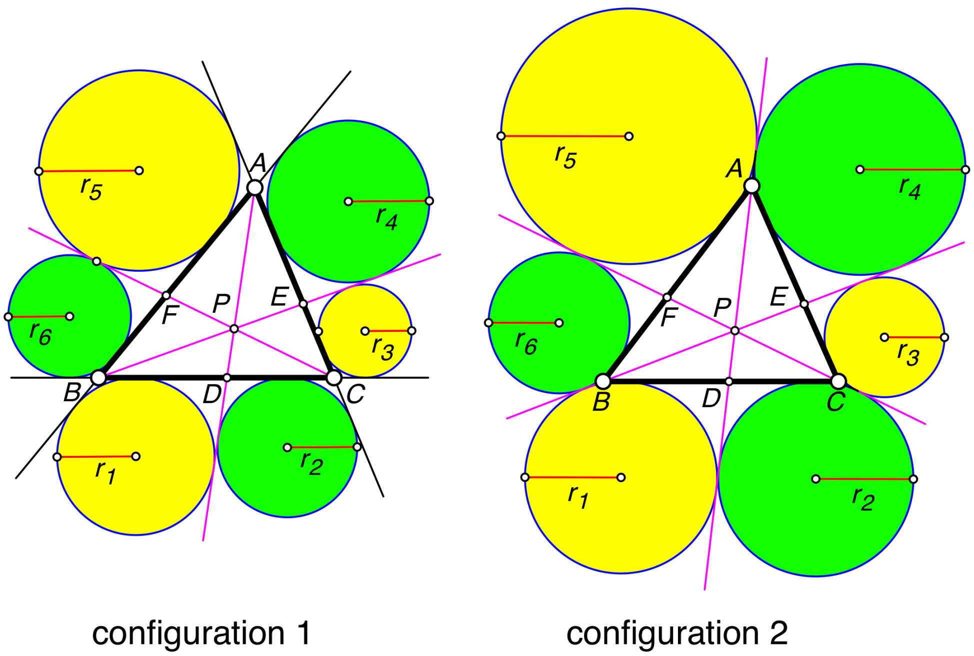

Figure 2. configurations

Notation: If X and Y are points, then we use the notation XY to denote either the line segment joining X and Y or the length of that line segment, depending on the context. The notation [XYZ] denotes the area of $\triangle XYZ$.

2. The Orthocenter

When P is the orthocenter of $\triangle ABC$, we have two results, depending upon which excircles are used.

Theorem 2.1. *Suppose the orthocenter, H, of $\triangle ABC$ lies inside $\triangle ABC$. Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through H situated as shown in Figure 3. Then $r_1r_3r_5 = r_2r_4r_6$.*

Proof. Note that the figure consisting of $\triangle ABD$ together with the circle with radius r_1 is similar to the figure consisting of $\triangle CBF$ together with the circle with radius r_6. Corresponding lengths in similar figures are in proportion, so

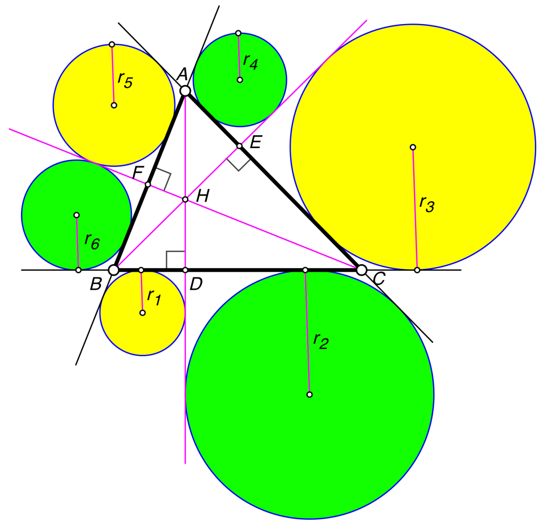

FIGURE 3. $r_1r_3r_5 = r_2r_4r_6$

$r_1/AD = r_6/CF$. Similarly, we find $r_3/BE = r_2/AD$ and $r_5/CF = r_4/BE$. Therefore

$$\frac{r_1}{r_6}\cdot\frac{r_3}{r_2}\cdot\frac{r_5}{r_4} = \frac{AD}{CF}\cdot\frac{BE}{AD}\cdot\frac{CF}{BE} = 1$$

which implies $r_1r_3r_5 = r_2r_4r_6$. □

A similar result occurs when different excircles are used.

Theorem 2.2. *Suppose the orthocenter, H, of $\triangle ABC$ lies inside $\triangle ABC$. Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through H situated as shown in Figure 4. Then $r_1r_3r_5 = r_2r_4r_6$.*

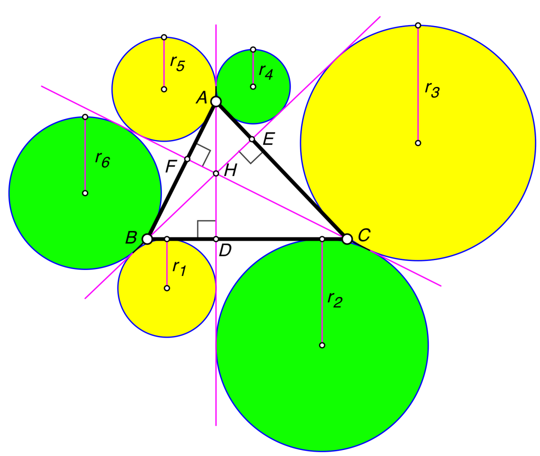

FIGURE 4. $r_1r_3r_5 = r_2r_4r_6$

Proof. Note that the figure consisting of $\triangle HBD$ together with the circle with radius r_1 is similar to the figure consisting of $\triangle HAE$ together with the circle with radius r_4. Corresponding lengths in similar figures are in proportion, so $r_1/BH = r_4/AH$. Similarly, we find $r_3/CH = r_6/BH$ and $r_5/AH = r_2/CH$. Therefore

$$\frac{r_1}{r_4} \cdot \frac{r_3}{r_6} \cdot \frac{r_5}{r_2} = \frac{BH}{AH} \cdot \frac{CH}{BH} \cdot \frac{AH}{CH} = 1$$

which implies $r_1 r_3 r_5 = r_2 r_4 r_6$. $\square$

3. The Centroid

Next, we will consider the case when the interior point P is the centroid. We start with a lemma.

Lemma 3.1. *Let M be the centroid of $\triangle ABC$ and let the medians be AD, BE, and CF. Label the segments BD, DC, CE, EA, AF, and FB with the numbers from 1 to 6 as shown in Figure 5. Let s_i be the semiperimeter of the triangle formed by the segment labeled i and the vertex of $\triangle ABC$ opposite that segment. Then*

$$s_1 + s_3 + s_5 = s_2 + s_4 + s_6. \tag{1}$$

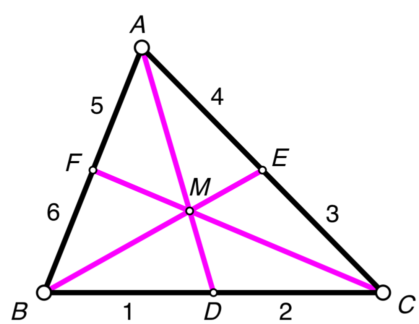

FIGURE 5. medians

Proof. We have the following six equations for the perimeters of the six triangles.

$$\begin{aligned} 2s_1 &= BD + AD + AB, & 2s_2 &= DC + AD + CA, \\ 2s_3 &= CE + BE + BC, & 2s_4 &= EA + BE + AB, \\ 2s_5 &= AF + CF + CA, & 2s_6 &= FB + CF + BC. \end{aligned}$$

Thus, $2s_1 + 2s_3 + 2s_5 - (2s_2 + 2s_4 + 2s_6) = (BD - DC) + (CE - EA) + (AF - FB) = 0$ and the lemma follows. $\square$

Recall the well-known formula for the length of the radius of an excircle. If the sides of a triangle have lengths a, b, and c, and the semiperimeter is s, then the radius of the excircle that touches the side of length a is $K/(s - a)$, where K is the area of the triangle. See, for example, [1, p. 79].

We can now state our results.

Theorem 3.1. *Let M be the centroid of $\triangle ABC$ and let the medians be AD, BE, and CF. Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through M situated as shown in Figure 6. Then*

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}.$$

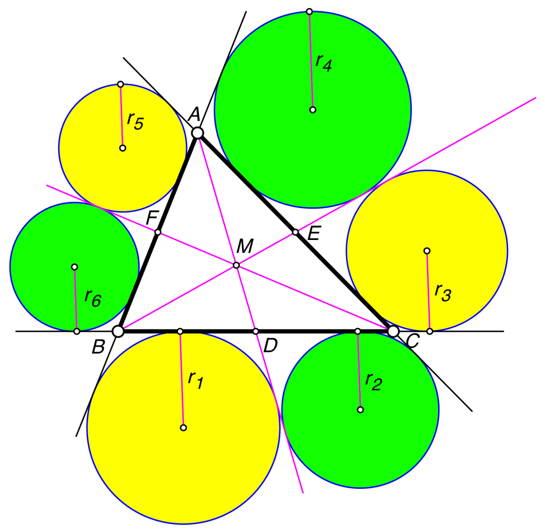

FIGURE 6.

Proof. The circle with radius r_i is an excircle of a triangle as shown in Figure 6. Let s_i be the semiperimeter of that triangle and let K_i be the area of that triangle. Note that each K_i is half the area of $\triangle ABC$. Denote this common value by K. From the formula for the length of the radius of an excircle, we have

$$\frac{1}{r_1}=\frac{s_1-BD}{K},\qquad \frac{1}{r_3}=\frac{s_3-CE}{K},\qquad \frac{1}{r_5}=\frac{s_5-AF}{K}.$$

Adding gives

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{s_1+s_3+s_5-s}{K}$$

where $s=BD+CE+AF$ is the semiperimeter of $\triangle ABC$. In the same manner, we find

$$\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}=\frac{s_2+s_4+s_6-s}{K}.$$

From Lemma 3.1, $s_1+s_3+s_5=s_2+s_4+s_6$. Thus,

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}$$

as required. □

A similar result occurs when different excircles are used.

Theorem 3.2. *Let M be the centroid of $\triangle ABC$ and let the medians be AD, BE, and CF. Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through M situated as shown in Figure 7. Then*

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}.$$

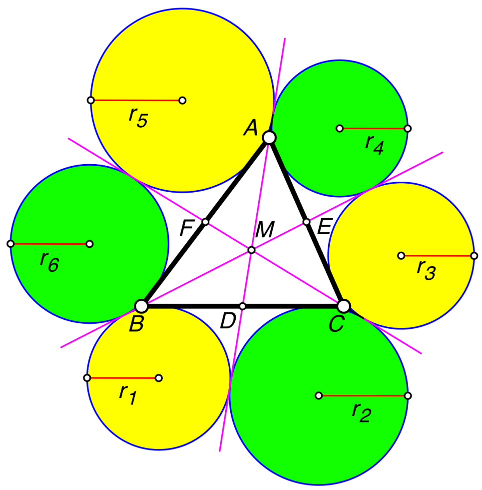

FIGURE 7.

Proof. The proof is similar to the proof of Theorem 3.1. In this case, the circles are excircles of triangles BMD, CMD, CME, AME, AMF, and BMF. These six triangles have the same area and their semiperimeters satisfy equation (1). The details are omitted. $\square$

4. THE GERGONNE POINT

Suppose the incircle of $\triangle ABC$ touches the sides BC, CA, and AB at points D, E, and F, respectively (Figure 8). Then the cevians AD, BE, and CF, meet at a point, G_e, known as the Gergonne Point of the triangle [1, p. 160].

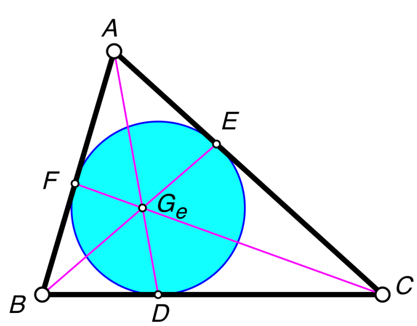

FIGURE 8. Gergonne Point

We will now find relationships between certain excircles associated with a triangle and the three cevians through its Gergonne Point. We start with a lemma.

Lemma 4.1. *Let D be the contact point of the incircle of $\triangle ABC$ with side BC (Figure 9). The excircle of $\triangle ABD$ that touches side BD has radius r_1. The excircle of $\triangle ADC$ that touches side DC has radius r_2. Then*

$$\frac{r_1}{r_2} = \frac{BD}{DC}.$$

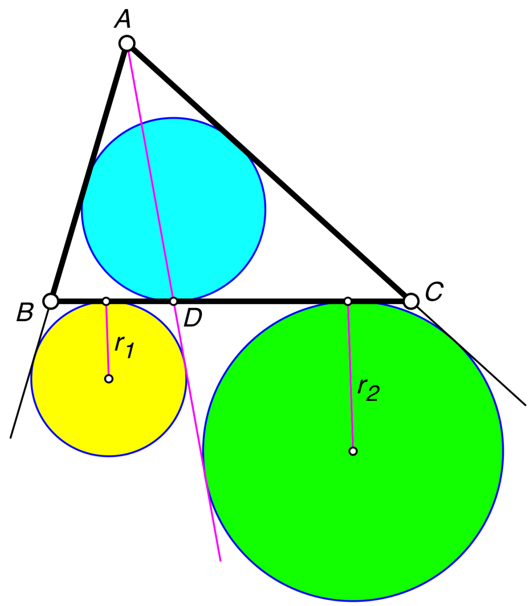

FIGURE 9. $\dfrac{r_1}{r_2} = \dfrac{BD}{DC}$

The following proof is due to Duca [2].

Proof. Using the formula for the radius of an excircle, we have

$$r_1 = \frac{2[ABD]}{AB + AD - BD} \quad \text{and} \quad r_2 = \frac{2[ADC]}{CA + AD - DC}. \tag{2}$$

By a well-known property of the incircle of a triangle [1, p. 87], $BD = s - CA$ and $DC = s - AB$, where $s = (AB + BC + CA)/2$. Thus $CA + BD = AB + DC$, which implies $AB + AD - BD = CA + AD - DC$. Therefore, the two denominators in equation (2) are equal. Hence $r_1/r_2 = [ABD]/[ADC]$. Since triangles ABD and ADC have the same altitude from A, the ratio of their areas will be proportional to the ratio of their bases. Consequently,

$$\frac{r_1}{r_2} = \frac{[ABD]}{[ADC]} = \frac{BD}{DC}.$$

□

We can now easily prove the following theorem.

Theorem 4.1. *Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through the Gergonne point situated as shown in Figure 10. Then $r_1 r_3 r_5 = r_2 r_4 r_6$.*

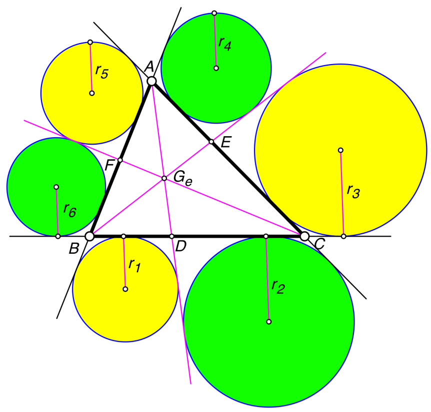

FIGURE 10. $r_1 r_3 r_5 = r_2 r_4 r_6$

Proof. By Lemma 4.1,

$$\frac{r_1}{r_2} \cdot \frac{r_3}{r_4} \cdot \frac{r_5}{r_6} = \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB}.$$

The expression on the right is equal to 1 by Ceva's Theorem. Thus, we have $r_1 r_3 r_5 = r_2 r_4 r_6$. □

5. THE NAGEL POINT

Suppose the excircles of $\triangle ABC$ touch the sides BC, CA, and AB at points D, E, and F, respectively, as shown in Figure 11. Then the cevians AD, BE, and CF meet at a point known as the Nagel Point of the triangle [1, p. 160].

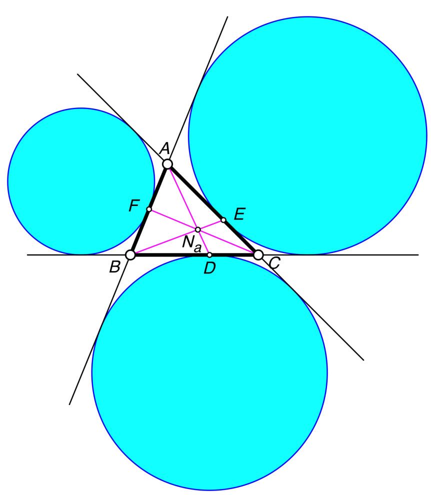

FIGURE 11. Nagel Point

Lemma 5.1. *Let N_a be the Nagel point of $\triangle ABC$. The cevians AD, BE, and CF through N_a divide $\triangle ABC$ into six small triangles numbered from 1 to 6 as shown in Figure 12. Let K_i be the area of triangle i. Then $K_1K_3K_5 = K_2K_4K_6$.*

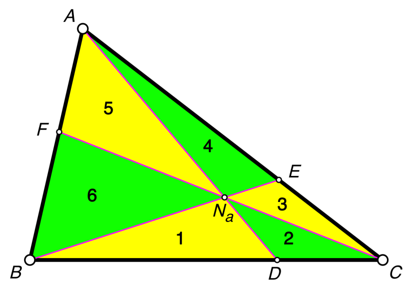

Figure 12. Nagel Point

Proof. This is a special case of Theorem 7.4 from [5]. □

Lemma 5.2. *Let N_a be the Nagel point of $\triangle ABC$. The cevians AD, BE, and CF through N_a divide $\triangle ABC$ into six small triangles numbered from 1 to 6 as shown in Figure 12. Let s_i be the semiperimeter of triangle i. Then $s_1s_3s_5 = s_2s_4s_6$.*

Proof. Let $a = BC$, $b = CA$, $c = AB$, and $s = (a + b + c)/2$. It is well-known that $AF = DC = s - b$, $FB = CE = s - a$, and $BD = EA = s - c$ [1, p. 88]. By Stewart's Theorem, we can compute the lengths of cevians AD, BE, and CF in $\triangle ABC$. We get

$$AD = \sqrt{s^2 - \frac{4K^2}{a(s-a)}}, \quad BE = \sqrt{s^2 - \frac{4K^2}{b(s-b)}}, \quad CF = \sqrt{s^2 - \frac{4K^2}{c(s-c)}},$$

where K is the area of $\triangle ABC$ and s is its semiperimeter. We can then use the Theorem of Menelaus on $\triangle ADC$ with traversal BE to find the ratio of AN_a to N_aD. We can use the same procedure to find how N_a divides the other cevians. We get

$$\frac{AN_a}{N_aD} = \frac{a}{s-a}, \quad \frac{BN_a}{N_aE} = \frac{b}{s-b}, \quad \frac{CN_a}{N_aF} = \frac{c}{s-c}.$$

These formulas can also be found in [4]. This allows us to compute the lengths of AN_a, N_aD, BN_a, N_aE, CN_a, and N_aF in terms of a, b, c, s, and K. This, in turn, gives us expressions for s_1, s_2, s_3, s_4, s_5, and s_6. Simplifying $s_1s_3s_5 - s_2s_4s_6$ using a computer algebra system shows that the result is 0. □

Theorem 5.1. *Let N_a be the Nagel point of $\triangle ABC$. The cevians AD, BE, and CF through N_a divide $\triangle ABC$ into six small triangles numbered from 1 to 6 as shown in Figure 12. Let r_i be the radius of the incircle of triangle i. Then $r_1r_3r_5 = r_2r_4r_6$.*

Proof. Using the notation from Lemmas 5.1 and 5.2, we have $r_i = K_i/s_i$, so

$$r_1r_3r_5 = \frac{K_1}{s_1}\frac{K_3}{s_3}\frac{K_5}{s_5} = \frac{K_2}{s_2}\frac{K_4}{s_4}\frac{K_6}{s_6} = r_2r_4r_6$$

as required. □

A similar result holds for excircles.

Lemma 5.3. *Let N_a be the Nagel point of $\triangle ABC$. The cevians AD, BE, and CF through N_a divide $\triangle ABC$ into six small triangles numbered from 1 to 6 as shown in Figure 12. Let s_i be the semiperimeter of triangle i. Then*

$$(s_1 - BD)(s_3 - CE)(s_5 - AF) = (s_2 - DC)(s_4 - EA)(s_6 - FB).$$

Proof. The proof is essentially the same as the proof of Lemma 5.2. Expressions for all the needed lengths have already been found in terms of a, b, c, s, and K. Simplifying $(s_1 - BD)(s_3 - CE)(s_5 - AF) - (s_2 - DC)(s_4 - EA)(s_6 - FB)$ using a computer algebra system shows that the result is 0. □

Theorem 5.2. *Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through the Nagel point situated as shown in Figure 13. Then $r_1r_3r_5 = r_2r_4r_6$.*

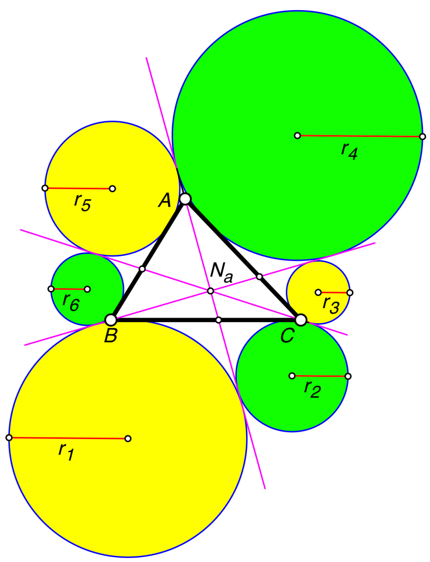

FIGURE 13. $r_1r_3r_5 = r_2r_4r_6$

Proof. The proof is the same as the proof of Theorem 5.1 only now $r_i = \dfrac{K_i}{(s - a_i)}$ where a_i is the length of the side of triangle i lying along a side of $\triangle ABC$. □

The results of Theorems 5.1 and 5.2 are so elegant that it is unlikely that they are true only because the complicated expressions found in the proofs of Lemmas 5.2 and 5.3 just happen to simplify to 0.

Open Question 1. *Are there simple proofs of Theorems 5.1 and 5.2 that do not involve a large amount of algebraic computation requiring computer simplification?*

6. Relationship Between Inradii and Exradii

We can make use of relationships between the radii of incircles associated with a figure to find relationships between the radii of associated excircles.

The following basic result was known to Japanese geometers of the Edo period as evidenced by the fact that it is equivalent to a problem found in the 1823 text, *Sangaku Shousen* [6], later printed as problem 4.3.3 in [3, p. 21].

Theorem 6.1 (Inradius/Exradius Invariant)**.** *Let A be a fixed point and let L be a fixed line that does not pass through A. Let B and C be variable points on L, with $B \neq C$. Let r be the inradius of $\triangle ABC$ and let r_a be the radius of the excircle that touches side BC (Figure 14). Then $\dfrac{1}{r} - \dfrac{1}{r_a}$ remains invariant as B and C vary along L.*

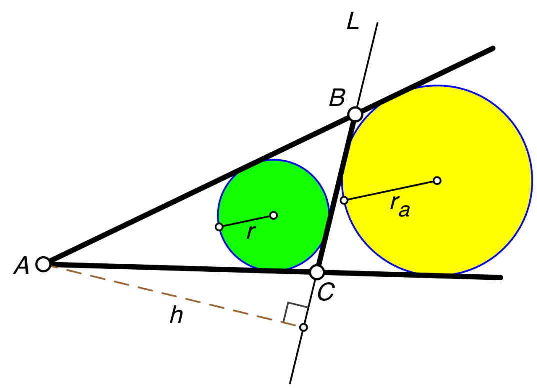

Figure 14. inradius and exradius

Proof. From the formulas for the length of an inradius and an exradius, we have

$$r = \frac{K}{s} \quad \text{and} \quad r_a = \frac{K}{s-a},$$

where a is the length of BC, K is the area of $\triangle ABC$, and s is its semiperimeter. Thus

$$\frac{1}{r} - \frac{1}{r_a} = \frac{s}{K} - \frac{s-a}{K} = \frac{a}{K} = \frac{2}{h},$$

where h is the distance from A to L. This proves the theorem since h remains fixed as B and C vary along L. □

The following result follows immediately.

Theorem 6.2 (Relationship Between Two Incircles and Two Excircles)**.** *Let AD be a cevian of $\triangle ABC$. Four circles are tangent to the sides of the triangle and the cevian as shown in Figure 15. Then*

$$\frac{1}{r_1} + \frac{1}{r_4} = \frac{1}{r_2} + \frac{1}{r_3}.$$

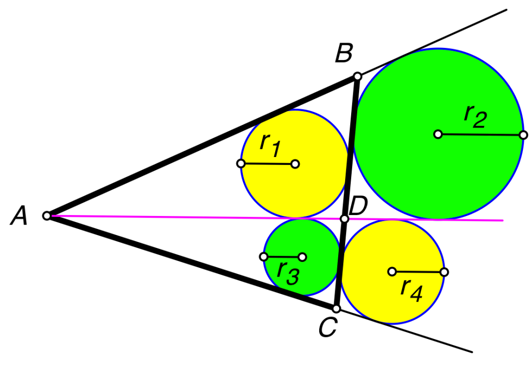

FIGURE 15. relationship between two incircles and two excircles

There is another nice relationship between inradii and exradii.

We start with a lemma.

Lemma 6.1 (Inradius/Exradius Fixed Angle Invariant). *Let A be a fixed point and let L_1 and L_2 be distinct fixed rays starting at A. Let B and C be variable points on L_1 and L_2, respectively, neither coinciding with A. Let r be the inradius of $\triangle ABC$ and let r_a be the radius of the excircle that touches side BC (Figure 16). Let K be the area of $\triangle ABC$. Then rr_a/K remains invariant as B and C vary.*

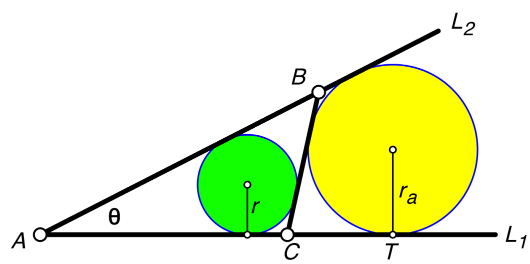

FIGURE 16. inradius and exradius

Proof. Let T be the contact point of the excircle with L_1. It is known that $AT = s$, where s is the semiperimeter of $\triangle ABC$ [1, Theorem 158]. Let $\angle BAC = \theta$. Note that the bisector of $\angle BAC$ passes through the centers of the two circles. Since $K = rs$, we have

$$\frac{rr_a}{K} = \frac{rr_a}{rs} = \frac{r_a}{s} = \frac{r_a}{AT} = \tan\frac{\theta}{2}.$$

Thus, rr_a/K is invariant because the angle θ is fixed. $\square$

Theorem 6.3 (Relationship Between Six Incircles and Six Excircles). *Let P be a point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six small triangles, named T_1 through T_6 as shown in Figure 17. Let r_i be the inradius of T_i. Let R_i be the exradius of T_i that touches a side of $\triangle ABC$. Then*

$$r_1r_3r_5R_1R_3R_5 = r_2r_4r_6R_2R_4R_6.$$

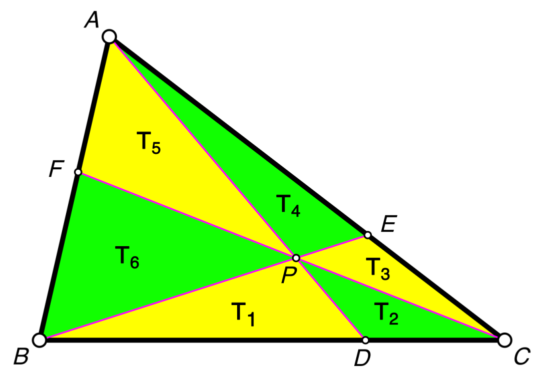

FIGURE 17. six triangles

Proof. Note that in Figure 17, AD and BE are straight lines passing through P, so $\angle BPD = \angle APE$. By Lemma 6.1, $r_1R_1/K_1 = r_4R_4/K_4$, with similar identities for the other two pairs of triangles. Therefore,

$$\frac{r_1R_1}{K_1}\cdot\frac{r_3R_3}{K_3}\cdot\frac{r_5R_5}{K_5}=\frac{r_4R_4}{K_4}\cdot\frac{r_3R_6}{K_6}\cdot\frac{r_2R_2}{K_2}.$$

But $K_1K_3K_5 = K_2K_4K_6$ by Theorem 7.4 from [5]. Thus, $r_1r_3r_5R_1R_3R_5 = r_2r_4r_6R_2R_4R_6$. □

Alternate formulation of Theorem 6.3: *In Figure 18, the product of the radii of the yellow circles is equal to the product of the radii of the green circles.*

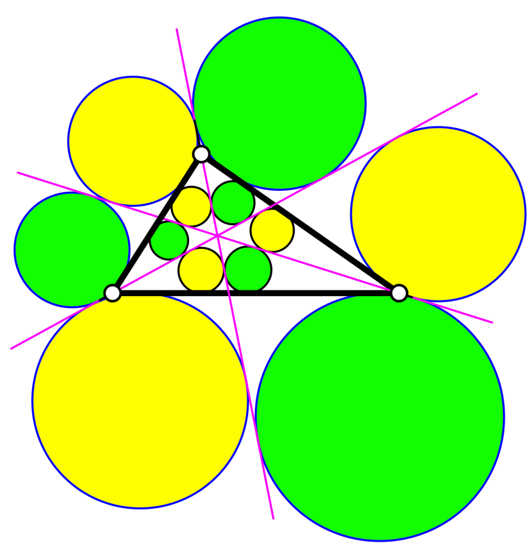

FIGURE 18. twelve circles

Theorem 6.3 provides an alternate proof to some earlier theorems. Applying Theorem 6.3 to Theorem 2.1 yields Theorem 2.2. Applying Theorem 6.3 to Theorem 5.1 yields Theorem 5.2.

We also have the following companion result, coloring the circles differently.

Consequence of Theorem 6.2: *In Figure 19, the sum of the reciprocals of the radii of the yellow circles is equal to the sum of the reciprocals of the radii of the green circles.*

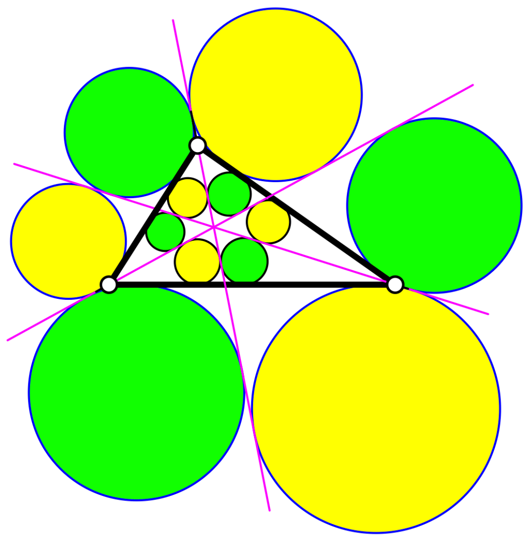

FIGURE 19. twelve circles

7. THE CIRCUMCENTER

The following theorem involving incircles was proven in [5].

Theorem 7.1. *Let O be the circumcenter of $\triangle ABC$. The cevians through O divide $\triangle ABC$ into six small triangles and circles are inscribed in these triangles as shown in Figure 20. The circle labeled i in the figure has radius r_i. Then*

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}.$$

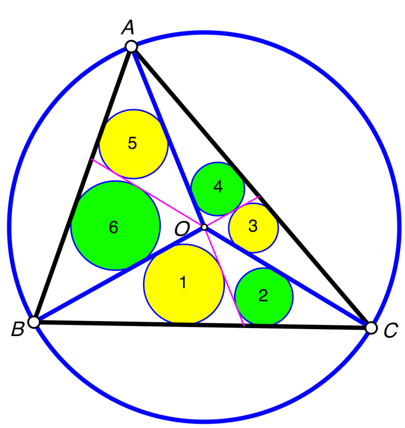

FIGURE 20. circumcenter with incircles

Applying Theorem 6.2 yields the following result about excircles.

Theorem 7.2. *Let O be the circumcenter of $\triangle ABC$ and let the cevians through O be AD, BE, and CF. Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through O situated as shown in Figure 21. Then*

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}.$$

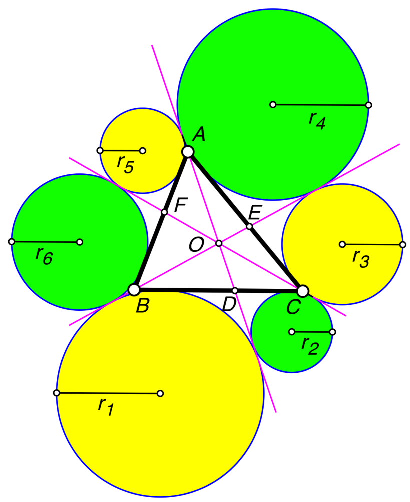

Figure 21. circumcenter with excircles

8. The Incenter

The following theorem involving incircles was proven in [5].

Theorem 8.1. *Let I be the incenter of $\triangle ABC$ and suppose $\angle ABC = 60°$. The cevians through I divide $\triangle ABC$ into six small triangles and circles are inscribed in these triangles as shown in Figure 22. The circle labeled i in the figure has radius r_i. Then*

$$\frac{1}{r_1}+\frac{1}{r_4}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_6}.$$

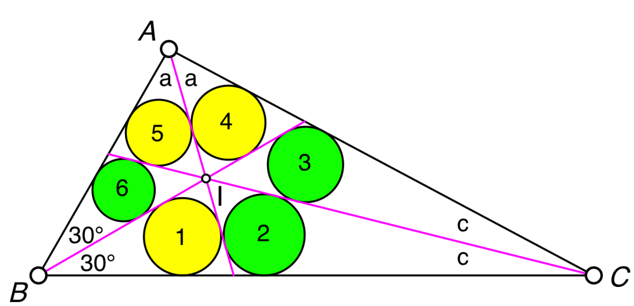

Figure 22. incenter with incircles

Applying Theorem 6.2 yields the following result about excircles.

Theorem 8.2. *Let I be the incenter of $\triangle ABC$ and suppose $\angle ABC = 60°$. Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through I situated as shown in Figure 23. Then*

$$\frac{1}{r_1}+\frac{1}{r_4}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_6}.$$

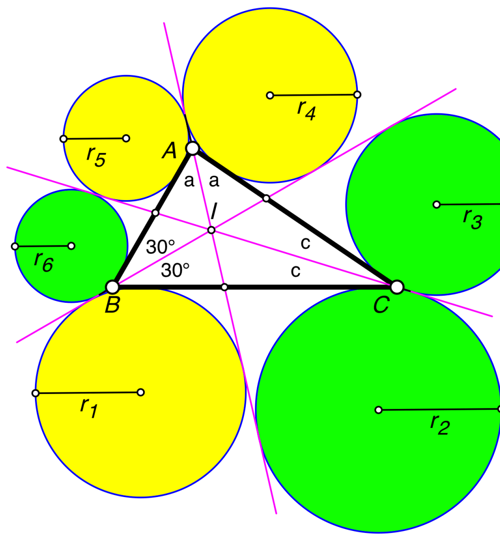

FIGURE 23. incenter with excircles

When we change $\angle ABC$ from 60° to 120°, we get a surprising result.

Theorem 8.3. *Let I be the incenter of $\triangle ABC$ and suppose $\angle ABC = 120°$. Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through I situated as shown in Figure 24. Then*

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4}+\frac{1}{r_6}=\frac{1}{r_2}+\frac{1}{r_5}.$$

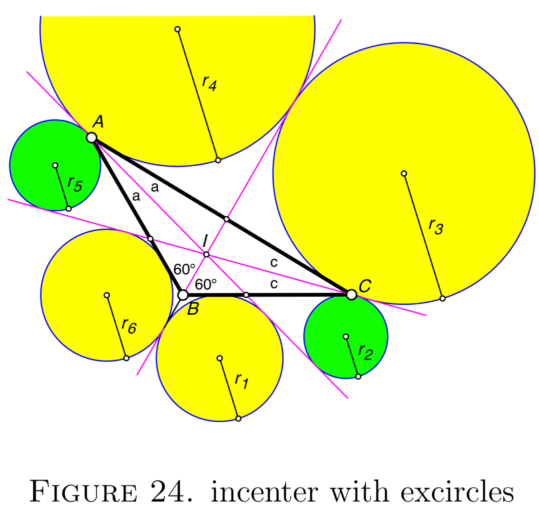

FIGURE 24. incenter with excircles

Proof. The proof is similar to the proof of Theorem 6.1 from [5]. Without loss of generality, we assume the circumradius of $\triangle ABC$ is $1/2$. Then we use the Law of Sines to find the lengths of all the line segments associated with $\triangle ABC$ and the three cevians in terms of the angles a and c. These values are given in [5]. Then noting that $c = 30° - a$, we use these values to compute the values of the r_i. Plugging these values into the expression

$$\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_4}+\frac{1}{r_6}-\frac{1}{r_2}-\frac{1}{r_5}$$

and simplifying (using a computer algebra system) gives 0. □

The proof of Theorem 8.2 depends on the proof of Theorem 8.1 whose only known proof (from [5]) involves computer simplification of complicated trigonometric expressions.

Open Question 2. *Are there simple proofs of Theorems 8.2 and 8.3 similar in complexity to the other proofs in this paper?*

In Theorems 8.2 and 8.3, $\angle ABC$ has a fixed value (60° or 120°). We can wonder what happens if we drop this restriction.

Theorem 8.4. *Let I be the incenter of $\triangle ABC$. Let r_1 through r_6 be the radii of six circles tangent to the sides of $\triangle ABC$ and the cevians through I situated as shown in Figure 25. If $\alpha = \frac{1}{2}\angle BAC$, $\beta = \frac{1}{2}\angle CBA$, $\gamma = \frac{1}{2}\angle ACB$, then*

$$\frac{\cos\gamma}{r_1}+\frac{\cos\alpha}{r_3}+\frac{\cos\beta}{r_5}=\frac{\cos\beta}{r_2}+\frac{\cos\gamma}{r_4}+\frac{\cos\alpha}{r_6}.$$

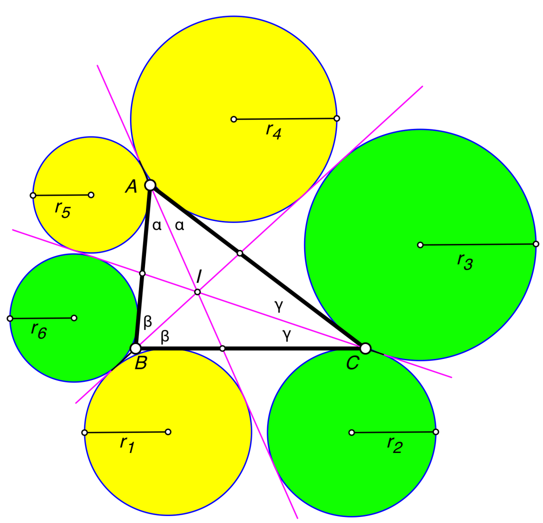

Figure 25. incenter with excircles

Proof. As with Theorem 8.3, the proof uses computer simplification of trigonometric expressions for the lengths of the segments in the figure. □

There can be many relationships between the r_i. Although Theorem 8.4 involves α, β, and γ, this does not preclude the existence of a relationship involving only the r_i.

Open Question 3. *Is there a simple relationship between the radii of excircles associated with an arbitrary triangle and the cevians through its incenter that does not depend on the shape of the triangle?*

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