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A sangaku problem involving four circles in a rectangle

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Abstract. We examine the figure of a sangaku problem involving four circles in a rectangle and show that the figure contains equilateral triangles and can naturally be embedded in a regular hexagon.

Keywords. equilateral triangle, regular hexagon

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1. Introduction

We consider the following problem in the sangaku hung in 1836 proposed by Onodera (小野寺倉吉定則) [2] (see Figure 1). We show that the figure of the problem contains equilateral triangles and can naturally be embedded in a regular hexagon.

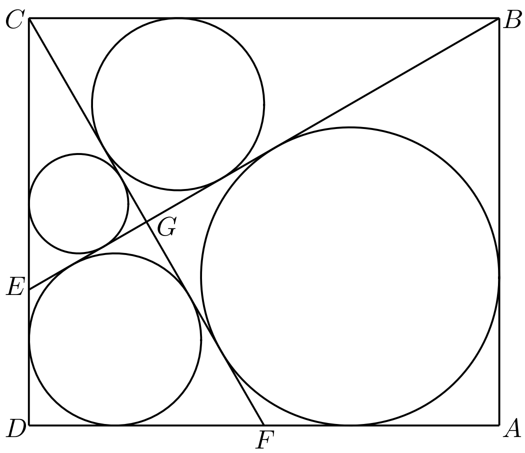

Figure 1.

Problem 1. For a rectangle $ABCD$, let E and F be points on the sides CD and DA, respectively such that the segments BE and CF meet in a point G. Show that if the triangles BCG and CFD have congruent incircles, and $ABGF$ and $DFGE$ are circumscribed, then the inradius of $ABGF$ equals the triple the inradius of the triangle CEG.

2. Solution

We examine the figure of the problem and show that the figure contains equilateral triangles and can naturally be embedded in a regular hexagon. We use the next proposition. An outline of the proof can be found in [1] (see Figure 2).

Proposition 1. *The line BE is the perpendicular bisector of CF.*

Proof. Obviously G is the midpoint of CF. Let H be the point of intersection of the lines BE and DA. We assume that the incircle of the triangle HFG has radius r and touches FH and GH at points V and W, respectively, where $|AB| = a$, $|GW| = s$, $|HV| = t$ and $|FV| = u$. Since $ABGF$ is circumscribed and $|BG| = |HW| + |GW| = s + t$, $|FG| = s + u$ and $|AF| = |DA| - |DF| = t + u - (r + u) = t - r$, we have $(s + t) + (t - r) = a + (s + u)$, i.e.,

$$u + r = 2t - a. \tag{1}$$

From the right triangle CFD, $2r = |CD| + |DF| - |CF| = a + (r + u) - 2(s + u)$. Therefore we have

$$r = a - 2s - u. \tag{2}$$

Eliminating r from (1) and (2), we get $a = s + t$, i.e., $|AB| = |GB|$. Similarly eliminating a from (1) and (2), we get $s + u = t - r$, i.e., $|GF| = |AF|$. Therefore $ABGF$ is a kite with a right angle at A. The proof is now complete. □

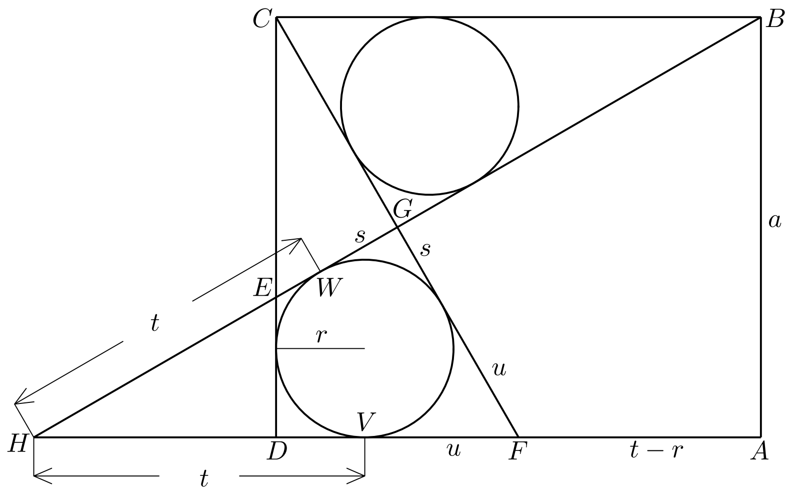

Figure 2.

Since $|CG| = |GF| = |DF|$ by the proposition, the triangles BCG and CFD are congruent. Hence $|BC| = |CF|$, while $|BC| = |BF|$. Therefore *the triangle BCF is equilateral.*

Let $|GE| = 1$. From $\angle GCE = 30°$, we have $|CG| = \sqrt{3}$ and $|AB| = |CE| + |DE| = 3$. Therefore the triangles CEG, BCG, HBA are similar with the ratio of similitude $1 : \sqrt{3} : 3$. This gives a solution of the problem.

Let P, Q and R be the incenters of $ABGF$, BCG and CFD, respectively (see Figure 3). Then the inradius of $ABGF$ equals $\sqrt{3}r$, where recall that r is the inradius of HFG, and $|QR| = 2|QG| = 2\sqrt{2}r$. While $|PQ| = \sqrt{PG^2 + QG^2} = \sqrt{(\sqrt{2}\sqrt{3}r)^2 + (\sqrt{2}r)^2} = 2\sqrt{2}r$. Similarly $|PR| = 2\sqrt{2}r$. Therefore *PQR is an equilateral triangle.*

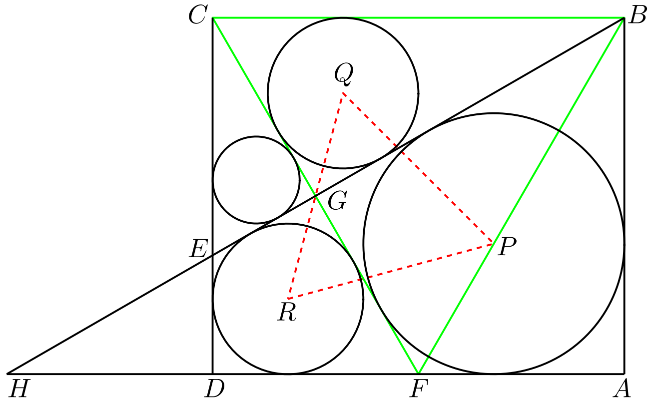

Figure 3.

Figure 4 shows that *each of the segments of the figure of Problem 1 coincides with a side or a part of a diagonal of a regular hexagon.*

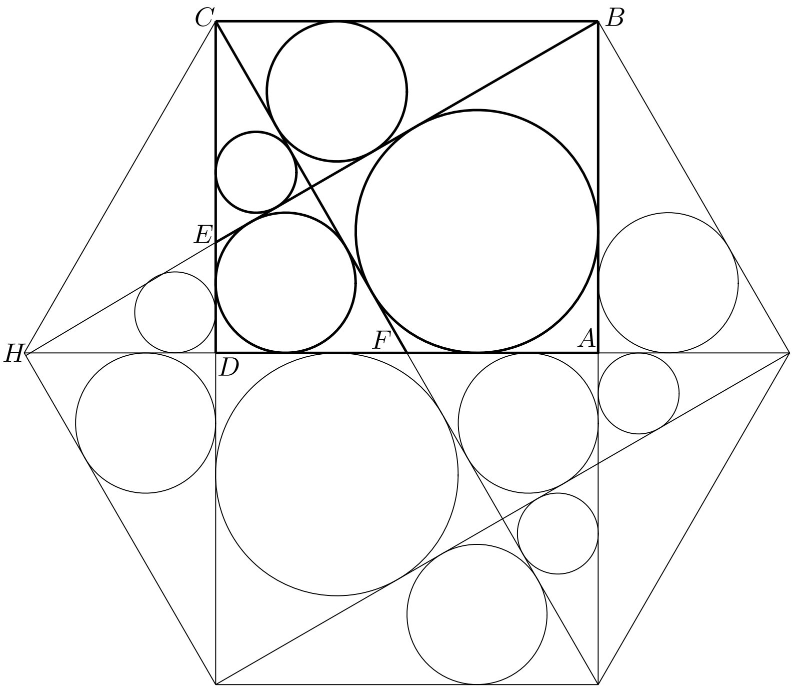

Figure 4.

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