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Remarks on Archimedean circles of Nagata and Ootoba

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Abstract. We consider two sangaku problems involving Archimedean circles proposed by Nagata and Ootoba, and give a brief historical note for their Archimedean circles. Nagata's Archimedean circles are obtained from the generalized Ootoba's Archimedean circle in [12] as a special case by division by zero [10]. We consider certain tangents of the Archimedean circle in the same case, which are not considered in [12].

Keywords. arbelos, thirdly discovered Archimedean circle, division by zero

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

We consider an arbelos consisting of three semicircles α, β and γ of diameters AO, BO and AB, respectively for a point O on the segment AB constructed on the same side of AB, where $|AO| = 2a$ and $|BO| = 2b$. The perpendicular to AB at O is called the axis. The axis divides the arbelos into two curvilinear triangles with congruent incircles, which are called the twin circles of Archimedes and have radius $r_{\mathrm{A}} = ab/(a+b)$ (see Figure 1). Circles of the same radius are said to be Archimedean [3].

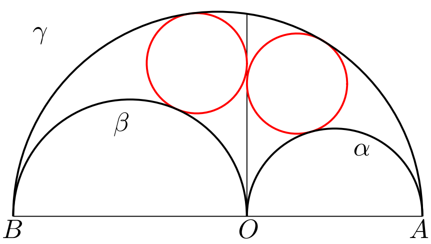

Figure 1.

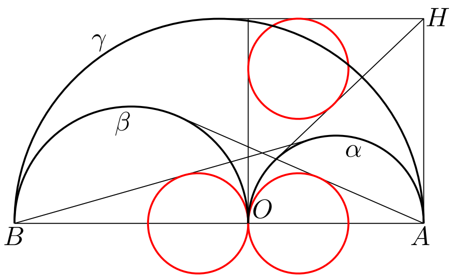

Figure 2.

In this paper, we consider two sangaku problems involving Archimedean circles (see Figure 2). The part (i) of the next problem was proposed by Nagata (永田岩三郎遵道) in the sangaku hung in 1831 at Atsuta Shrine (熱田神宮) in Nagoya [8]. The part (ii) was proposed by Ootoba (大鳥羽源吉守敬) in the sangaku hung in 1853 at Takenobu Inari Shrine (武信稲荷神社) in Kyoto [9] .

Problem 1. (i) Show that the circle touching α at O internally and the tangent of β from A is congruent to the circle touching β at O internally and the tangent of α from B.
(ii) Let H be the point of intersection of the tangent of γ parallel to AB and the tangent of α at A. Show that the incircle of the triangle made by the tangent of γ parallel to AB, the axis, and the remaining tangent of α from H is Archimedean.

We have generalized Ootoba's Archimedean circle in [12], and have also shown that Nagata's Archimedean circles can be obtained as a special case of the generalization using the recently made definition of division by 0 [10]. However neither the tangent of α and the Archimedean circle nor the tangent of γ and the Archimedean circle have been considered in the special case. In this article we consider the tangents in that case. Another generalization of Ootoba's Archimedean circle can be found in [18]. A brief historical note for Archimedean circles is also given.

2. Thirdly discovered Archimedean circle

There are several problems involving the twin circles of Archimedes in Wasan geometry as in [1], [4], [6]. On the other hand, it is very rare to find a problem involving other Archimedean circles. While the two congruent circles in Problem 1(i) are Archimedean and are also considered in [3]. Therefore the two problems in Problem 1 are involving Archimedean circles different from the twin circles of Archimedes.

Leon Bankoff found an Archimedean circle in 1974 [2], which has been believed as the thirdly discovered Archimedean circle. But the Archimedean circles in Problem 1 are dated in 1831 and 1853, which were considered earlier than Bankoff circle. However Nagata's problem can be solved without expressing the common radius of the two circles in terms of a and b. Thereby we can not be sure that Nagata noticed that the common radius equaled r_{A} in his problem. While it is easy to see that the common radius equals r_{A} by the similarity. In any case, it seems to be appropriate that Nagata's Archimedean circles should be recorded in the history of studying Archimedean circles together with Ootoba's Archimedean circle.

Nagata's problem is stated in a slightly generalized form in [5], and a generalization of this can be found in [19]. Nagata's Archimedean circles are also generalized to generalized arbeloi called the collinear arbelos and the skewed arbelos in [17].

3. Generalized Ootoba's Archimedean circles and division by zero

In this section we use the definition in [10]:

$$a/0 = 0 \quad \textit{for any real number } a. \tag{1}$$

We use a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a). We consider the next theorem, which is a generalization of Ootoba's Archimedean circle in Problem 1(ii) [12] (see Figure 3).

Theorem 1. *If $H(\neq A)$ is a point on the half line perpendicular to AB with initial point A constructed on the same side of AB as γ, then the incircle of the triangle made by the remaining tangents of α and γ from H and the axis is Archimedean, and the radius of the excircle of the triangle touching the axis from the side opposite to H equals b.*

We consider the tangents of α and γ from H in the theorem in the case $A = H$. Let H be as in the theorem and $h = |AH|$. Since the centers of the Archimedean circle and the excircle in the theorem have coordinates

$$\left(r_{\mathrm{A}}, \frac{a^2}{h} + \frac{hr_{\mathrm{A}}}{a}\right), \quad \left(-b, \frac{a(a+b)}{h}\right),$$

respectively [12], they touch at O if $h = 0$ by (1). In this case we denote the two circles by δ_A and β_A, respectively (see Figure 4). The circle δ_A coincides with Nagata's Archimedean circles touching α internally.

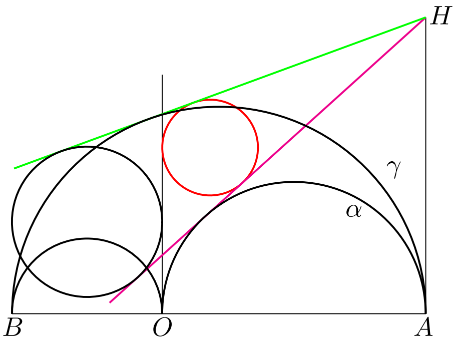

Figure 3.

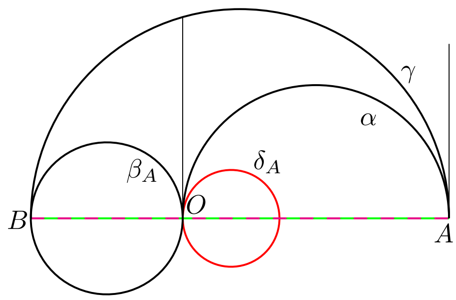

Figure 4.

Since the remaining tangent of α from H touches α at the point of coordinates

$$\left(\frac{2a^3}{a^2+h^2}, \frac{2a^2h}{a^2+h^2}\right),$$

it is expressed by the equation

$$f_\alpha(x,y) = a^2(2a-x) - 2ayh + xh^2 = 0.$$

Considering $f_\alpha(x,y)/h^i = 0$ for $i = 0, 1, 2$, we get $x = 2a$, $y = 0$, $x = 0$ if $h = 0$ by (1), where we exclude the lines $x = 2a$ and $x = 0$, because the former does not touch δ_A and the latter does not pass through A. Notice that x-axis ($y = 0$) touches the circle δ_A, β_A and γ, since $\tan(\pi/2) = 0$ by (1). Hence the tangent from H to α coincides with the x-axis in the case $H = A$.

Since the remaining tangent of γ from H touches γ at the point of coordinates

$$\left(\frac{2a(a+b)^2 - 2bh^2}{(a+b)^2+h^2}, \frac{2(a+b)^2h}{(a+b)^2+h^2}\right),$$

it is expressed by the equation

$$f_\gamma(x,y) = (a+b)^2(2a-x) - 2(a+b)yh + (x+2b)h^2 = 0.$$

Considering $f_\gamma(x,y)/h^i = 0$ for $i = 0, 1, 2$, we get $x = 2a$, $y = 0$, $x = -2b$ if $h = 0$ by (1), where we exclude the lines $x = 2a$ and $x = -2b$, because they do not touch the circle δ_A. Hence the tangent from H to γ also coincides with the x-axis if $H = A$. Therefore we can conclude that *the tangents of γ and α from H*

coincide with the x-axis in the case $H = A$ (see Figure 4). A similar phenomenon can be found in [15].

The remaining Nagata's Archimedean circle touching β internally is obtained by exchanging the roles of the points A and B. For an extensive introduction of division by zero see [20], and its application to Wasan geometry see [7], [11], [13, 14, 16], [21].

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