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Solution to 2017-1 Problem 4 with division by zero

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Abstract. We consider a sangaku problem involving four tangent circles and show a simple application of the recent definition of division by zero to Wasan geometry.

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Mathematics Subject Classification (2010). 01A20, 51M04

1. Introduction

In this article we consider 2017-1 Problem 4 [12], in which the text of the problem was lost. Figure 1 shows the existing figure. Following to the custom of Wasan geometry to consider the relationships between the radii of circles, we guess that the problem asks to find the relationship of the radii of the four circles, where two circles are congruent as in the figure. Thereby we consider the following problem:

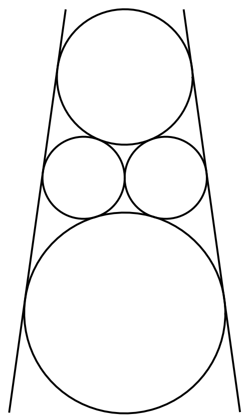
Figure 1.

Problem 1. Let α_1 and α_2 be circles of radius a touching externally. A circle β of radius b touches α_1 and α_2 externally and a circles γ of radius c touch α_1 and α_2 externally from the another side. If the three circles α_i $(i = 1, 2)$, β and γ share the external common tangent, find a in terms of b and c.

2. SOLUTION

We denote the external common tangent of α_i, β and γ by s_i. Let A_i be the center of α_i and let T be the point of tangency of α_1 and s_1. We denote the angle between the vectors $\overrightarrow{A_2A_1}$ and $\overrightarrow{A_1T}$ by θ. There are four cases to be considered: $0 \le \theta < \pi/2$, $\theta = \pi/2$, $\pi/2 < \theta < \pi$ and $\theta = \pi$ (see Figures 2, 3, 4, 5). The same problem considering the case $0 \le \theta < \pi/2$ can be found in [1, 2, 3, 4, 11, 13, 14], where the contents of [2] and [14] are almost the same and an integer solution $(a, b, c) = (16, 17, 68)$ is given in [4].

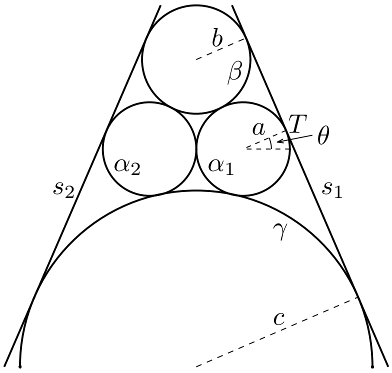

Figure 2: $0 \le \theta < \pi/2$

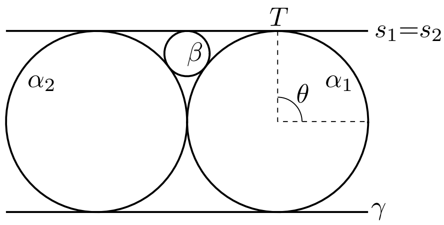

Figure 3: $\theta = \pi/2$

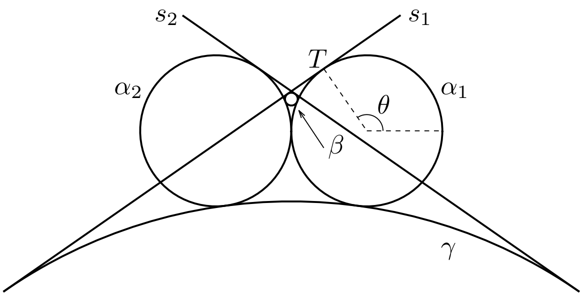

Figure 4: $\pi/2 < \theta < \pi$

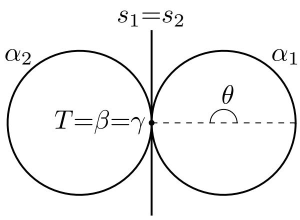

Figure 5: $\theta = \pi$

Theorem 1. *The following statements are true.*
(i) *If* $0 \le \theta < \pi/2$*, we get*

$$a = \frac{4bc}{b + 6\sqrt{bc} + c}. \tag{1}$$

(ii) *If* $\pi/2 < \theta < \pi$*, we get*

$$a = \frac{4bc}{b - 6\sqrt{bc} + c}. \tag{2}$$

Proof. We prove (ii) (see Figure 6). Assume that O is the point of intersection of the lines s_1 and s_2, B is the center of β and $r = |OT|$. Then $\angle BOT = 2\angle A_1OT$, while $t_1 = \tan \angle BOT = -b/(2\sqrt{ab} - r)$ and $t_2 = \tan \angle A_1OT = a/r$. Substituting the last two equations in $t_1 = 2t_2/(1 - t_2^2)$, we have

$$a^2b - 4a\sqrt{ab}r + (2a - b)r^2 = 0. \tag{3}$$

If we invert the figure in the circle with center O and radius r, the point of tangency of s_1 and β is the inverse of the point of tangency of s_1 and γ. Therefore we get $(2\sqrt{ab}-r)(2\sqrt{ac}-r)=r^2$. This implies

$$r=\frac{2\sqrt{abc}}{\sqrt{b}+\sqrt{c}}. \tag{4}$$

Substituting (4) in (3), we get (2). The part (i) is proved similarly. □

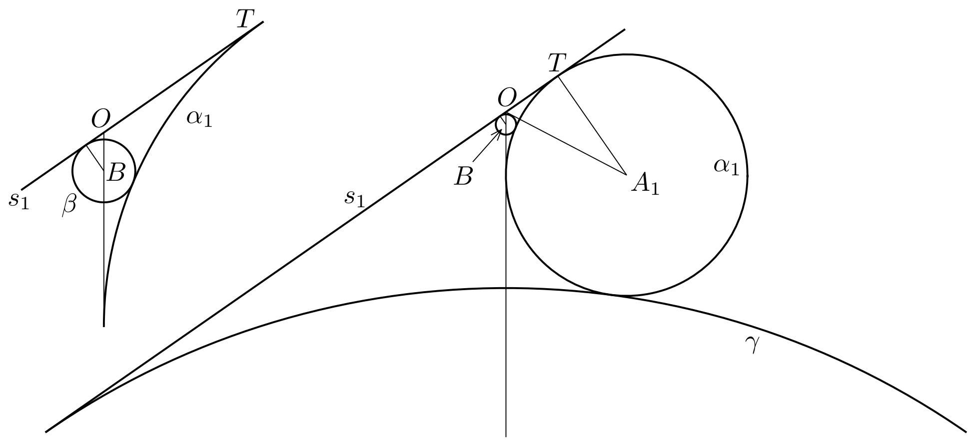

Figure 6: $\pi/2 < \theta < \pi$

3. THE CASES $\theta = \pi/2$ AND $\theta = \pi$

In this section we consider the cases $\theta = \pi/2$ and $\theta = \pi$ by assuming the definition of the devision by zero: $z/0 = 0$ for any real number z [5]. For a very brief introduction to the definition of division by zero, see [6]. Since a line has curvature 0, its radius equals $1/0 = 0$.

Let us consider the case $\theta = \pi/2$ (see Figure 3). In this case we get $a = 4b$ and $c = 0$ since γ is a line, as just noted above. Hence (1) does not hold. Meanwhile if we divide the numerator and the denominator of the right sides of (1) by c, we get

$$a=\frac{4b}{\frac{b}{c}+6\sqrt{\frac{b}{c}}+1}. \tag{5}$$

Then $a = 4b$ and $c = 0$ satisfy (5). Therefore (5) *expresses the relationships of the three radii in the case* $0 \le \theta \le \pi/2$.

Let us consider the case $\theta = \pi$ (see Figure 5). In this case β and γ are point circles. Therefore we get $b = c = 0$, which do not satisfy (2). However if we express the relation between the three radii in the form

$$a(b-\sqrt{bc}+c)=4bc, \tag{6}$$

then $b = c = 0$ satisfy (6). Therefore (6) *expresses the relationships of the three radii in the case* $\pi/2 < \theta \le \pi$.

For more applications of the definition of division by zero to Wasan geometry and related areas, see recent publications [7, 8, 9, 10].

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Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.

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