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# Solution to 2017-3 Problem 5

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**Abstract.** 2017-3 Problem 5 is generalized.

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## 1. Introduction

For a triangle $EFG$, let $\gamma_1$, $\gamma_2$, $\cdots$, $\gamma_n$ be circles of radius $r$ such that they touch the side $EF$ from the inside of $EFG$, $\gamma_1$ and $\gamma_2$ touch, $\gamma_i$ ($i = 3, 4, \cdots, n$) touches $\gamma_{i-1}$ from the side opposite to $\gamma_1$, $\gamma_1$ touches the side $GE$, $\gamma_n$ touches the sides $FG$ (see Figure 1). In this case we say that *$EFG$ has $n$ circles of radius $r$ on $EF$* [1]. Those are a variety of circles called congruent circles on a line [2]. In this paper we generalize the following problem (see Figure 2).

**Problem 1** (2017-3 Problem 5)**.** Let $ABCD$ be a rectangle with center $O$ and circumcircle $\gamma$ of radius $s$. A circle of radius $r$ touches the side $BC$ and the minor arc of $\gamma$ cut by $BC$ at each of the midpoints. If $OAB$ has two circles of radius $r$ on $AB$, find $s/r$.

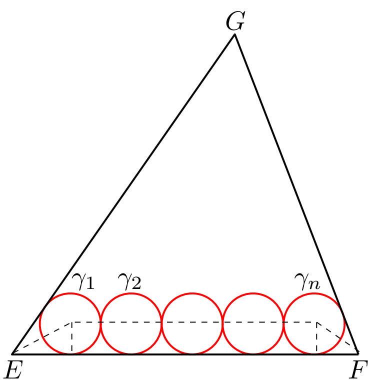

Figure 1.

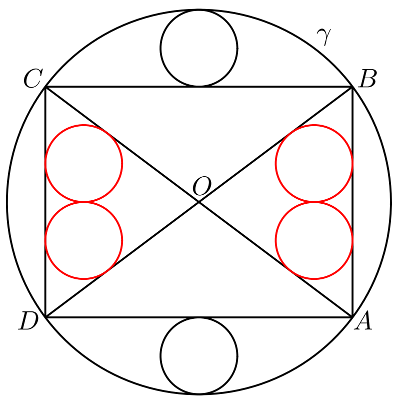

Figure 2.

## 2. Generalization

Notice that $EFG$ has $n$ circles of radius $r$ on $EF$ if and only if

$$|EF| = 2(n-1)r + r\cot(\angle GEF/2) + r\cot(\angle GFE/2).$$

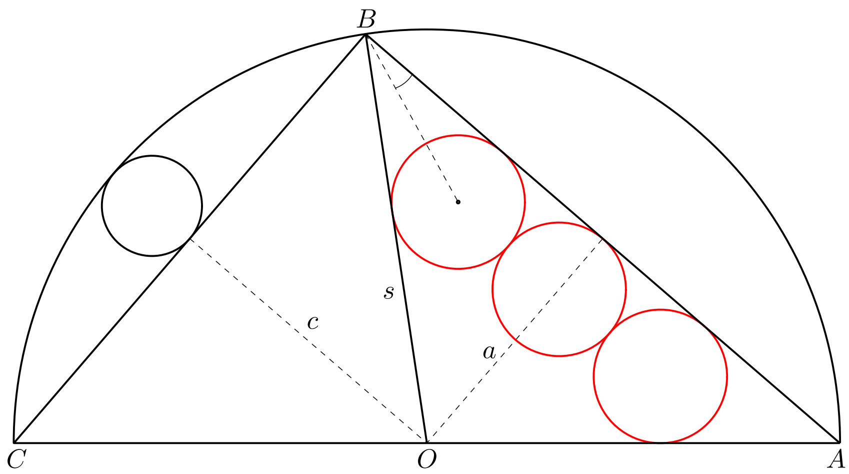


Figure 3: $n = 3$

**Theorem 1.** *Let $ABC$ be a right triangle with hypotenuse $CA$ and circumcircle $\gamma$ of radius $s$ center $O$. Assume that a circle of radius $r'$ touches $BC$ and the minor arc $BC$ of $\gamma$ at each of the midpoints, $OAB$ has $n$ circles of radius $r$ on $AB$, and $t = \cot(\angle ABO/2)$. Then $r = r'$ if and only if*

$$t = \frac{1}{2}\left(1 + \sqrt{4n+1}\right). \tag{1}$$

*Proof.* Let $a = |BC|/2$, $c = |AB|/2$ (see Figure 3). Obviously we have

$$s = c + 2r'. \tag{2}$$

From $\tan(\angle ABO) = a/c$, we get $t = \left(c + \sqrt{a^2+c^2}\right)/a = (c+s)/a$, i.e.,

$$at = c + s. \tag{3}$$

The power of the midpoint of $BC$ with respect to the circle $\gamma$ equals

$$2r'(c+s) = a^2. \tag{4}$$

Since $OAB$ has $n$ circles of radius $r$ on $AB$, we have

$$c = (n-1)r + rt. \tag{5}$$

Then eliminating $a$, $c$, $s$ from (2), (3), (4), (5), we get

$$\frac{r'}{r} = \frac{1}{t+1} + \frac{n}{t^2-1}.$$

This implies

$$1 - \frac{r'}{r} = \frac{1}{t^2-1}\left(t - \frac{1+\sqrt{4n+1}}{2}\right)\left(t - \frac{1-\sqrt{4n+1}}{2}\right).$$

Therefore $r' = r$ and (1) are equivalent, since $t > 1$. $\square$

If $r = r'$ in the theorem, we denote the figure by $\mathcal{S}(n)$. Theorem 1 shows that the configuration $\mathcal{S}(n)$ can be constructed uniquely for a positive integer $n$. Problem 1 asks to find $s/r$ for $\mathcal{S}(2)$. The notations for $\mathcal{S}(n)$ used in Theorem 1 will be used throughout this paper.

From (1), (2), (5), we get the following corollary, which is a generalization of Problem 1:

**Corollary 1.** *The following relation holds for* $\mathcal{S}(n)$.

$$\frac{s}{r} = n + 1 + \frac{1}{2}\left(1 + \sqrt{4n+1}\right). \tag{6}$$

Also (3) and (4) yield $a = 2r't$. Hence we have the following relation for $\mathcal{S}(n)$:

$$a = 2rt. \tag{7}$$

The equation (1) shows that $t$ equals the golden number $\left(1+\sqrt{5}\right)/2$ for $\mathcal{S}(1)$. Therefore the incenter of the triangle $OAB$, the foot of perpendicular from the incenter to $BC$, the point $B$, and the midpoint of $AB$ are the vertices of a golden rectangle for $\mathcal{S}(1)$ (see Figure 4).

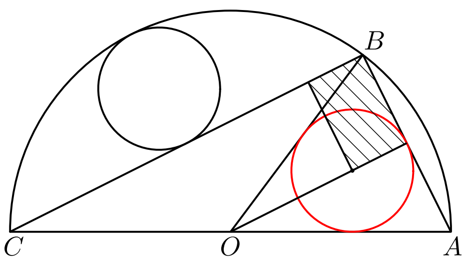

Figure 4: $\mathcal{S}(1)$

Since $s/r = n + 1 + t$ by (1) and (6) for $\mathcal{S}(n)$, $s/r$ is an integer if and only if $t$ is an integer for $\mathcal{S}(n)$.

**Corollary 2.** *For* $\mathcal{S}(n)$, $s/r$ *is an integer if and only if there is a positive integer* $k$ *such that* $n = k(k+1)$. *In this event* $t = k+1$ *and* $s/r = (k+1)^2 + 1$.

*Proof.* $s/r$ is an integer if and only if $4n+1$ is a square of an odd integer by (6). This is equivalent to $4n+1 = (2k+1)^2$ for some positive integer $k$. In this event $n = k(k+1)$. The rest of the corollary follows from (1) and (6). □

Notice that Problem 1 is the case $k = 1$. Figures 5, 6 show the cases $k = 2, 3$.

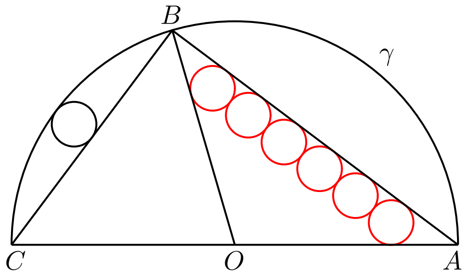

Figure 5: $\mathcal{S}(6)$, $k = 2$, $s = 10r$

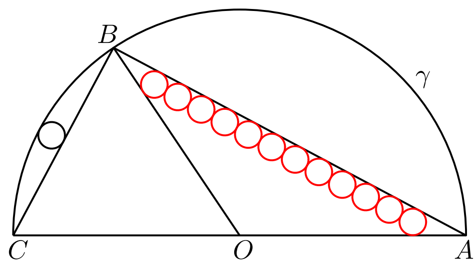

Figure 6: $\mathcal{S}(12)$, $k = 3$, $s = 17r$

## 3. Special case

We have $|BC|/|AB| = 4/3$ for $\mathcal{S}(2)$ by (1), (5) and (7), while $|BC|/|AB| = 3/4$ for $\mathcal{S}(6)$. Hence the right triangles $ABC$ in $\mathcal{S}(2)$ and $\mathcal{S}(6)$ are 3-4-5 triangles. Therefore $\mathcal{S}(2)$ and $\mathcal{S}(6)$ can be constructed from the same triangle (see Figure 7). Then there arises a problem to determine all the such right triangles each of which derives $\mathcal{S}(n)$ and $\mathcal{S}(m)$ for some positive integers $n$ and $m$. However we show that there is no other such triangles except the one just mentioned.

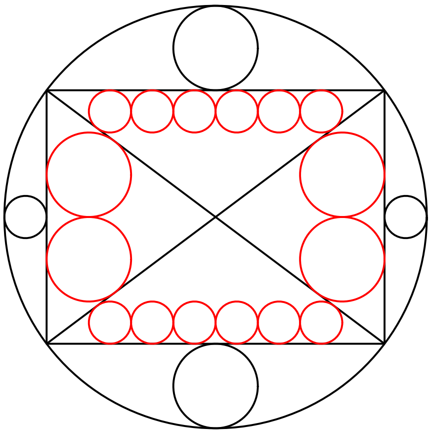

Figure 7: $\mathcal{S}(2)$ and $\mathcal{S}(6)$

**Theorem 2.** *$\mathcal{S}(2)$ and $\mathcal{S}(6)$ are only the pair which can be derived from the same right triangle.*

*Proof.* For $\mathcal{S}(n)$, let us assume that a circle of radius $r'$ touches $AB$ and the minor arc $AB$ of $\gamma$ at the midpoints, $OBC$ has $m$ circles of radius $r'$ on $BC$, and $t' = \cot(\angle BCO/2)$. Then we have (1) and

$$t' = \frac{1}{2}(1 + \sqrt{4m+1}) \tag{8}$$

by Theorem 1. Since $(\angle ABO)/2 + (\angle BCO)/2 = 45°$, $(t-1)(t'-1) = 2$ holds. Substituting (1) and (8) in the last equation and rearranging, we have

$$m^2n^2 - 10mn - 4(m+n) + 8 = 0. \tag{9}$$

The positive integer solutions of (9) are $(n, m) = (2, 6), (6, 2)$ for $n \leq 13$. Let us assume $n > 13$. Solving (9) for $m$ we get

$$m = \frac{5n + 2 \pm (n+2)\sqrt{4n+1}}{n^2}.$$

However $(5n+2)^2 - \left((n+2)\sqrt{4n+1}\right)^2 = -4(n-2)n^2 < 0$. Therefore we get

$$m = m(n) = \frac{5n + 2 + (n+2)\sqrt{4n+1}}{n^2}.$$

Then $m(n)$ is a monotonically decreasing function of $n$, and $m(14) < 1$. Therefore (9) has no positive integer solutions for $n > 13$. □

## 4. Open problem

If we divide each of the isosceles triangles $OAB$ and $OBC$ in $\mathcal{S}(2)$ by the perpendicular bisectors of $AB$ and $BC$, we get four congruent 3-4-5 triangles. Therefore if $OAB$ has two circles of radius $r$ on $AB$, $OBC$ has also two circles of radius $r$ on $BC$ (see Figure 8). In general if $OAB$ has $n$ circles of radius $r$ on $AB$ in $\mathcal{S}(n)$ and $OBC$ has $m$ circles of radius $r$ on $BC$, we denote the figure by $\mathcal{S}(n, m)$. Now we can say that $\mathcal{S}(2, 2)$ exists.

For $\mathcal{S}(6)$, let us assume that $AB$ has 6 circles of radius $r$ on $AB$. Then $t = 3$ by (1). Hence $t' = \cot(\angle BCO/2) = 1 + 2/(t-1) = 2$ and $|BC|/2 = 2rt = 6r = (5-1)r + rt'$ by (7). Therefore $OBC$ has 5 circles of radius $r$ on $BC$, i.e., $\mathcal{S}(6, 5)$ exists (see Figure 9). However the problem to determine all the existing $\mathcal{S}(n, m)$ remains unsolved. Notice that both $\mathcal{S}(2, 2)$ and $\mathcal{S}(6, 5)$ are also made from 3-4-5 triangles.

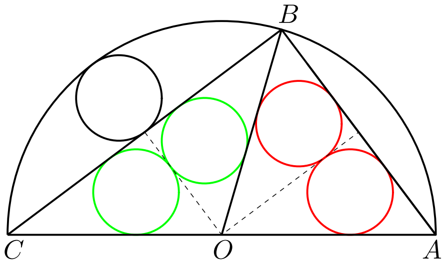

Figure 8: $\mathcal{S}(2, 2)$

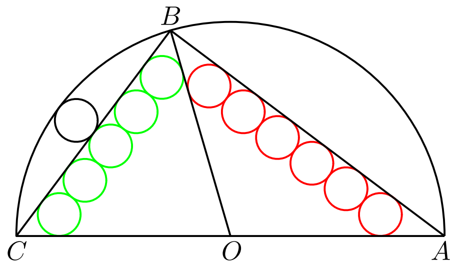

Figure 9: $\mathcal{S}(6, 5)$

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[1] H. Okumura, A note on an isosceles triangle containing a square and three congruent circles, Sangaku J. Math., **1** (2017) 24–34.

[2] H. Okumura, Configurations of congruent circles on a line, Sangaku J. Math., **1** (2017) 24–34.