Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 1 (2017), pp. 39–41
Received 1 November 2017. Published on-line 11 November 2017
web: http://www.sangaku-journal.eu/

A note on the problems involving congruent circles in Tenzan Kaitei

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: okmr@protonmail.com

Problems involving congruent circles in Tenzan Kaitei are considered.

Keywords. congruent circles

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

In this note we consider two problems in [2]. Figure 1 is taken from the 156th problem, which consists of eight small congruent circles, two medium congruent circles, a large circles and a rectangle. The problem is asking to find the ratio of the radii of the large circle and the small circles. Therefore the two medium congruent circles and the two small congruent circles touching them internally are unnecessary, because the ratio is uniquely determined without them. However it seem that the proposer of the problem considered to demonstrate his or her discovery of the existence of the two medium circles and the two small circles by the figure.

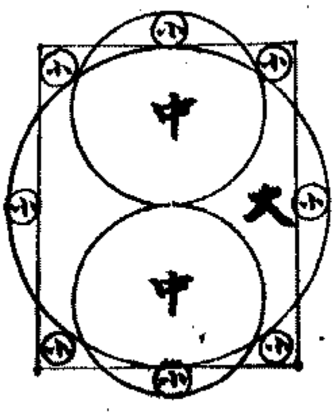

Figure 1.

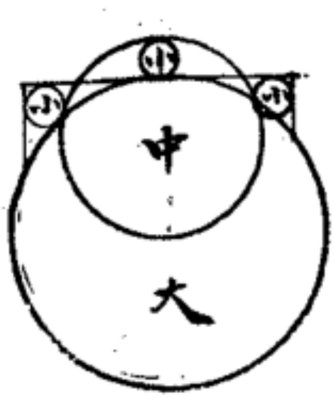

Figure 2.

Figure 2 is taken from the 157th problem in the same book, which consists of three congruent small circles, one medium circle, one large circle and three segments. The answer says that the radius of the large circle equals twice the difference of the radii of the medium circle and the small circles, which is equivalent to that the medium circle passes through the center of the large circle.

2. Theorem

The assertion of the figure of the 156th problem and the 157th problem may be summarized as follows (see Figures 3):

Theorem 1. *Let α and β be circles touching externally with common internal tangent t. Let β' be the circle congruent to β touching α externally and t from the same side as α. Let δ be a circle touching β internally at the point farthest from t. Then δ passes through the center of α if and only if δ touches β' externally.*

Proof. Let A, B, D and a, b, d be the centers and the radii of α, β', δ, respectively. Assume that δ passes through A. If F is the foot of perpendicular from B to the line AD, we get $d = a/2 + b$, $|BF| = 2\sqrt{ab}$ and $|DF| = |3b - d|$. Therefore δ and β' touch externally, since

$$|BD| = \sqrt{|BF|^2 + |DF|^2} = \sqrt{4ab + (3b - d)^2} = a/2 + 2b = b + d.$$

The converse holds by the uniqueness of the figure. □

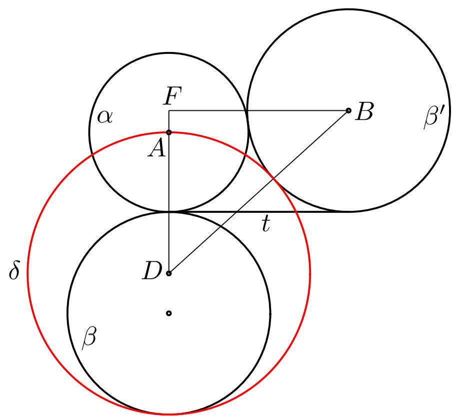

Figure 3.

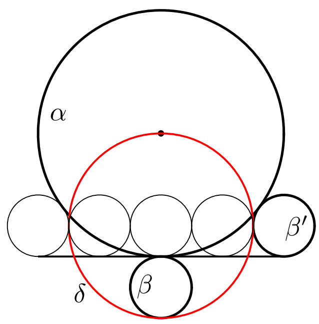

Figure 4.

In the proof, the sign of $3b - d$ depends on the value of a/b, and $3b - d = 0$ holds if and only if $a/b = 4$. In this event the circles α and β' are the center circle and one of the sides of the configuration $\mathcal{A}(5)$ in [1] (see Figure 4).

3. Configurations arising from the theorem

Let P be the point of tangency of α and β, and let R be the reflection of P in the center of α (see Figure 5). Let δ' be the circle of radius d' touching α internally at R passing through the center of β. The powers of P with respect to δ and δ' are $-2ab$. Therefore t is the radical axis of δ and δ' and passes through their common points. Let Q be the point of tangency of β' and t. The circle congruent to α touching β externally and t from the same side as β touches t at Q. The circle also touches δ' externally by the theorem. Figure 6 is constructed from Figure 5, which consists of two segments, two circles of radius a, two circles of radius b, four circles of radius d, and four circles of radius d'.

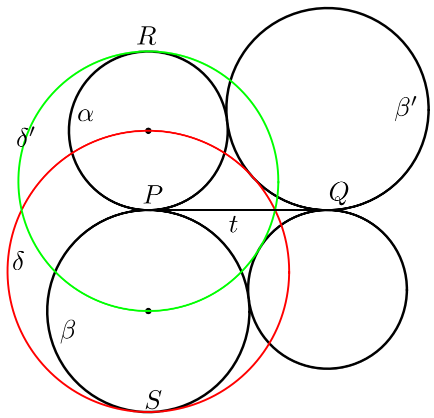

Figure 5.

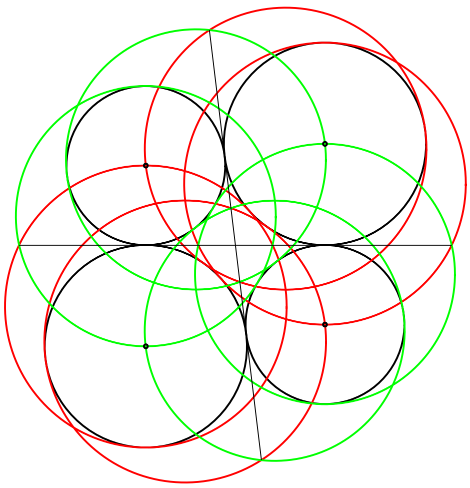

Figure 6.

Let S be the reflections of P in the centers of β. Then the triangles PQR and PSQ are similar, since $|PQ|/|PR| = |PS|/|PQ| = \sqrt{b/a}$. Therefore $\angle SQR$ is a right angle, and Q lies on the circle with diameter RS, which is denoted by γ. Figure 7 is obtained by adding the circle γ and three circles congruent to γ to Figure 6.

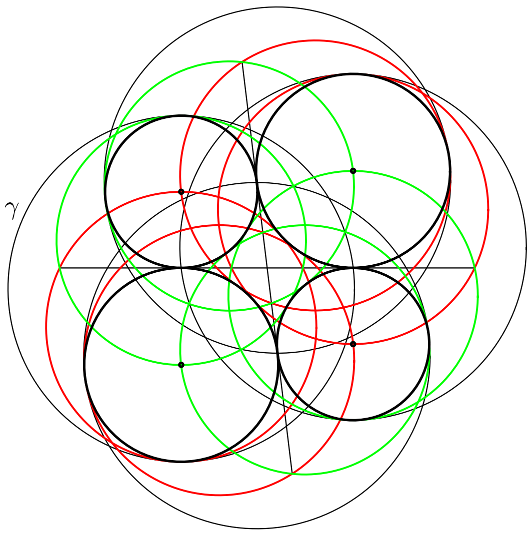

Figure 7.

REFERENCES

[1] H. Okumura, Configurations of congruent circles on a line, Sangaku J. Math., **1** (2017) 24–34.

[2] Tenzan Kaitei (点竄階梯), Tohoku University Wasan Material Database, `http://www.i-repository.net/il/meta_pub/G0000398wasan_4100006710`.