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# An Additional Property of Problem 3 and A Solution to Problem 4 of Problems 2023-1

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**Abstract.** We will discuss an additional property to problem 3 and illustrate a solution to problem 4 given in [1].

**Keywords.** Square configuration, Fibonacci number, Complex number.

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## 1. Introduction

In this paper, we will discuss an additional property to problem 3 and illustrate a solution to problem 4 given in [1].

## 2. An Additional Property of Problem 3

**Problem 3.** *The following are squares, where vertices lie counterclockwise in these orders: $ABCD$, $BEFG$, $JDHI$, $CGKH$, $IKLM$, $JMON$, $OLPQ$. (see Figure 1). Say something interesting for this figure.*

A solution to this problem is given by V. R. Shrimali in [2]. He showed

$$OL^2 = a_6^2 = 64a_0^2 + 25a_1^2 - 80a_0a_1\cos\alpha.$$

In general, if $a_n$ is the side of the $n^{th}$ $(n \geq 2)$ square, he obtained

$$a_n^2 = (F_{n-1}a_0)^2 + (F_{n-2}a_1)^2 - 2(F_{n-1}a_0)(F_{n-2}a_1)\cos\alpha,$$

where $|AD| = a_0$, $|BE| = a_1$, $\angle CBG = \alpha$, and $F_n$ is the $n^{th}$ Fibonacci Number such that $F_0 = F_1 = 1$.

We will discuss an additional property of this square configuration that also involves Fibonacci Numbers. We will approach this problem using complex numbers. Bold small letters represent complex numbers and '$\mathbf{i}$' denotes the unit imaginary number called "iota".

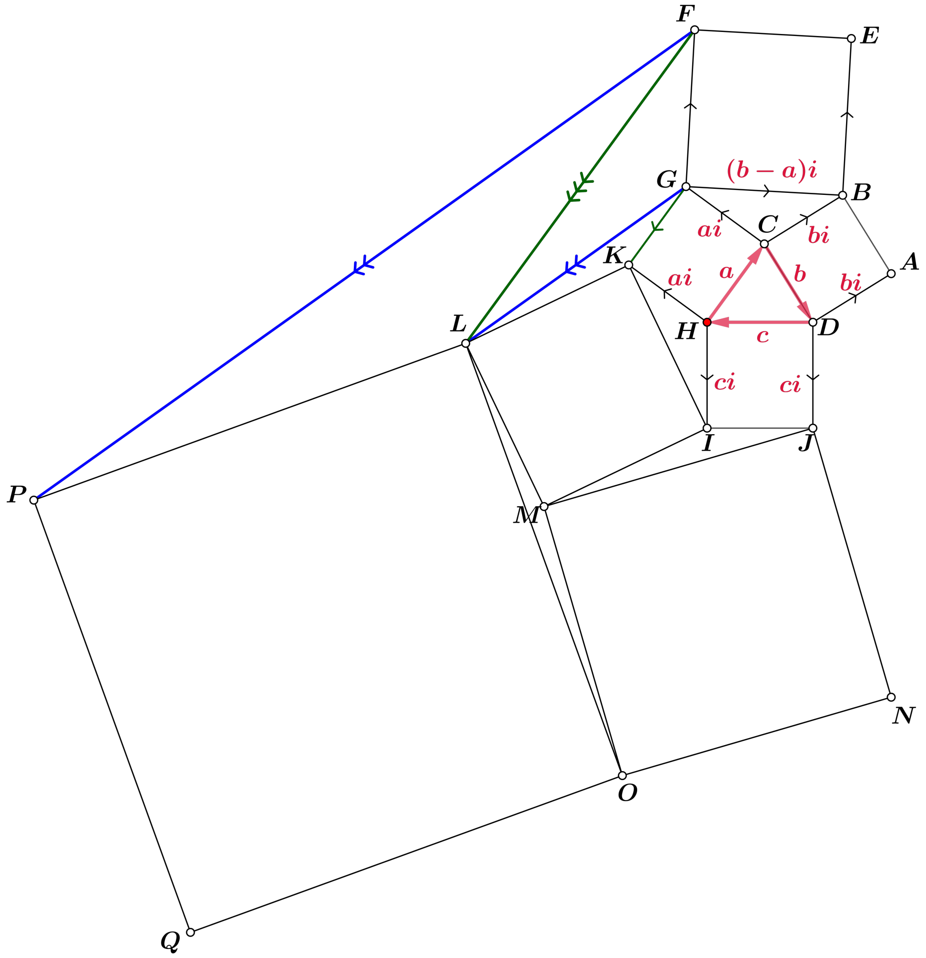


FIGURE 1.

Let $H$ be the origin, $\overrightarrow{HC} = \mathbf{a}$, $\overrightarrow{CD} = \mathbf{b}$, and $\overrightarrow{DH} = \mathbf{c}$. Hence $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Now we can find complex representations for the remaining points as given below. Note that $\overrightarrow{CB}$ can be obtained by rotating $\overrightarrow{CD}$ counterclockwise through an angle of $90°$ with respect to the point $C$. Therefore, $\overrightarrow{CB} = \overrightarrow{CD}\mathbf{i} = \mathbf{bi}$. So with respect to origin (H), the complex representation for $B \equiv \mathbf{a} + \mathbf{bi}$. Similarly, we obtain

$$A \equiv \mathbf{a} + \mathbf{b} + \mathbf{bi}, \quad G \equiv \mathbf{a} + \mathbf{ai}, \quad E \equiv \mathbf{2a} - \mathbf{b} + \mathbf{bi}, \quad F \equiv \mathbf{2a} - \mathbf{b} + \mathbf{ai},$$
$$K \equiv \mathbf{ai}, \quad I \equiv \mathbf{ci}, \quad J \equiv \mathbf{a} + \mathbf{b} + \mathbf{ci}, \quad M \equiv \mathbf{c} - \mathbf{a} + \mathbf{ci}, \quad L \equiv \mathbf{c} - \mathbf{a} + \mathbf{ai},$$
$$N \equiv \mathbf{a} + \mathbf{b} - (\mathbf{4a} + \mathbf{3b})\mathbf{i}, \quad O \equiv -(\mathbf{2a} + \mathbf{b}) - (\mathbf{4a} + \mathbf{3b})\mathbf{i},$$
$$Q \equiv -(\mathbf{7a} + \mathbf{4b}) - (\mathbf{4a} + \mathbf{3b})\mathbf{i}, \quad P \equiv -(\mathbf{7a} + \mathbf{4b}) + \mathbf{ai}.$$

Observe that $\overrightarrow{FL} = (\mathbf{c} - \mathbf{a} + \mathbf{ai}) - (\mathbf{2a} - \mathbf{b} + \mathbf{ai}) = -\mathbf{4a} = 4\overrightarrow{GK}$.
Therefore,

$$GK \parallel FL \quad \text{and} \quad |GK| = \frac{1}{4}|FL|. \quad \text{Assume } f_1 = \frac{1}{4}.$$

Again,

$$\overrightarrow{GL} = (\mathbf{c} - \mathbf{a} + \mathbf{ai}) - (\mathbf{a} + \mathbf{ai}) = -(\mathbf{3a} + \mathbf{b})$$

and

$$\overrightarrow{FP} = (-(\mathbf{7a} + \mathbf{4b}) + \mathbf{ai}) - (\mathbf{2a} - \mathbf{b} + \mathbf{ai}) = -\mathbf{3}(\mathbf{3a} + \mathbf{b}).$$

Therefore,

$$GL \parallel FP \quad \text{and} \quad |GL| = \frac{1}{3}|FP|. \quad \text{Assume } f_2 = \frac{1}{3}.$$

We define the top left point such as $P$, the '*farthest point*'. Then by drawing squares continuously following this pattern, we can generate as many as the '*farthest point*' as we want. If the next few successive '*farthest points*' are $X$, $Y$, $Z$, $W$, etc., and the distances of these points are measured from $G$ and $F$ then by similar calculations, we obtain

$$GP \parallel FX \text{ and } |GP| = \tfrac{4}{11}|FX|, \qquad GX \parallel FY \text{ and } |GX| = \tfrac{3}{8}|FY|,$$
$$GY \parallel FZ \text{ and } |GY| = \tfrac{11}{29}|FZ|, \qquad GZ \parallel FW \text{ and } |GZ| = \tfrac{8}{21}|FW|,$$

where $f_3 = \frac{4}{11}$, $f_4 = \frac{3}{8}$, $f_5 = \frac{11}{29}$, $f_6 = \frac{8}{21}$ and this pattern will be continued.
If we '*redefine*' Fibonacci numbers: $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n > 1$, then the sequence of fractions

$$f_1 = \frac{1}{4},\ f_2 = \frac{1}{3},\ f_3 = \frac{4}{11},\ f_4 = \frac{3}{8},\ f_5 = \frac{11}{29},\ f_6 = \frac{8}{21}, \cdots$$

contain Fibonacci numbers, where $f_n$ denotes the $n^{th}$ fraction.
Observe that

$$f_1 = \frac{F_0 + F_2}{F_2 + F_4},\ f_3 = \frac{F_2 + F_4}{F_4 + F_6},\ f_5 = \frac{F_4 + F_6}{F_6 + F_8}, \cdots,$$

and

$$f_2 = \frac{F_2}{F_4},\ f_4 = \frac{F_4}{F_6},\ f_6 = \frac{F_6}{F_8}, \cdots.$$

Interestingly, both the sequences $\{f_1, f_3, f_5, \cdots\}$ and $\{f_2, f_4, f_6, \cdots\}$ will converge to $1/\phi^2$, where $\phi = (\sqrt{5}+1)/2$ is the golden ratio. Let's prove it formally.

If $f_{2n}$ denotes the $n$-th term of the sequence $\{f_2, f_4, f_6, \cdots\}$ then $f_{2n} = \dfrac{F_{2n}}{F_{2n+2}}$.

Using the identity $F_m F_{n+1} + F_{m-1} F_n = F_{m+n}$ [3], taking $m = n$ and $m = n+2$, we may write

$$f_{2n} = \frac{F_{2n}}{F_{2n+2}} = \frac{F_n F_{n+1} + F_{n-1} F_n}{F_{n+2} F_{n+1} + F_{n+1} F_n} = \frac{1 + \frac{F_{n-1}}{F_{n+1}}}{\frac{F_{n+2}}{F_n} + 1}. \tag{1}$$

Using the identity $\lim\limits_{n\to\infty} \dfrac{F_{n+m}}{F_n} = \phi^m$ [3], from (1), we get

$$\lim_{n\to\infty} f_{2n} = \lim_{n\to\infty} \frac{F_{2n}}{F_{2n+2}} = \frac{1 + \frac{1}{\phi^2}}{1 + \phi^2} = \frac{1}{\phi^2}. \tag{2}$$

If $f_{2n-1}$ denotes the $n$-th term of the sequence $\{f_1, f_3, f_5, \cdots\}$ then

$$f_{2n-1} = \frac{F_{2n-2} + F_{2n}}{F_{2n} + F_{2n+2}} = \frac{\frac{F_{2n-2}}{F_{2n}} + 1}{1 + \frac{F_{2n+2}}{F_{2n}}}.$$

Using (2), we obtain

$$\lim_{n\to\infty} f_{2n-1} = \frac{\frac{1}{\phi^2} + 1}{1 + \phi^2} = \frac{1}{\phi^2}.$$

Therefore,

$$\lim_{n\to\infty} f_{2n} = f_{2n-1} = \frac{1}{\phi^2}.$$

Also,

$$\lim_{n\to\infty} \frac{f_2 \cdot f_4 \cdots f_{2n}}{f_1 \cdot f_3 \cdots f_{2n-1}} = \lim_{n\to\infty} \frac{F_{2n} + F_{2n+2}}{F_{2n+2}} = 1 + \frac{1}{\phi^2}.$$

## 3. Solution to Problem 4

**Problem 4.** *Show $AB \perp CD$ and $|AB| = |CD|$ for Figure 2.*

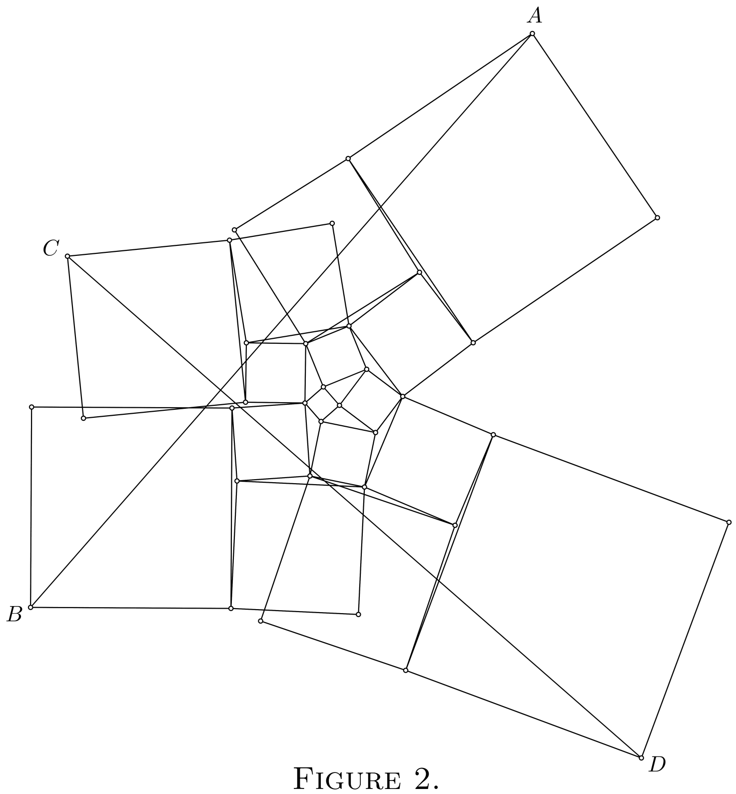


FIGURE 2.

**Solution.** We will approach this problem using complex numbers as in the previous problem. By removing some outer squares from the original configuration, we labeled the vertices from $E$ to $W$ as shown in Figure 3.

Let $O$ be the origin, $\overrightarrow{OL} = \mathbf{a}$, $\overrightarrow{LK} = \mathbf{b}$, and $\overrightarrow{KO} = \mathbf{c}$. Hence $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

Again, we have $\overrightarrow{ON} = \mathbf{ai} = \overrightarrow{LM}$, $\overrightarrow{LF} = \mathbf{bi} = \overrightarrow{KG}$, $\overrightarrow{KJ} = \mathbf{ci} = \overrightarrow{OP}$.

Now, we can write complex representations for all points w.r.t $O$ as shown below.

We have $L \equiv \mathbf{a}$, $K \equiv \mathbf{a} + \mathbf{b}$, $N \equiv \mathbf{ai}$, $M \equiv \mathbf{a} + \mathbf{ai}$, $F \equiv \mathbf{a} + \mathbf{bi}$, $G \equiv \mathbf{a} + \mathbf{b} + \mathbf{bi}$, $J \equiv \mathbf{a} + \mathbf{b} + \mathbf{ci}$, $P \equiv \mathbf{ci}$. So, $\overrightarrow{PN} = (\mathbf{a} - \mathbf{c})\mathbf{i}$. Then $\overrightarrow{PS} = \overrightarrow{NT} = \mathbf{a} - \mathbf{c}$.

Thus $T \equiv (\mathbf{c} - \mathbf{a}) + \mathbf{ai}$, $S \equiv (\mathbf{c} - \mathbf{a}) + \mathbf{ci}$, and $\overrightarrow{JS} = (\mathbf{c} - \mathbf{b} - 2\mathbf{a})$.

Again, $\overrightarrow{JQ} = \overrightarrow{SR} = (\mathbf{c} - \mathbf{b} - 2\mathbf{a})\mathbf{i}$. Therefore, $Q \equiv \mathbf{a} + \mathbf{b} + \mathbf{ci} + (\mathbf{c} - \mathbf{b} - 2\mathbf{a})\mathbf{i}$ and $R \equiv (\mathbf{c} - \mathbf{a}) + \mathbf{ci} + (\mathbf{c} - \mathbf{b} - 2\mathbf{a})\mathbf{i} = -(2\mathbf{a} + \mathbf{b}) - (4\mathbf{a} + 3\mathbf{b})\mathbf{i}$. ($\because \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$)

Since $\overrightarrow{GJ} = (\mathbf{c} - \mathbf{b})\mathbf{i}$ then $\overrightarrow{GH} = \mathbf{b} - \mathbf{c}$. Hence $H \equiv \mathbf{a} + \mathbf{b} + \mathbf{bi} + (\mathbf{b} - \mathbf{c}) = (2\mathbf{a} + 3\mathbf{b}) + \mathbf{bi}$. ($\because \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$)

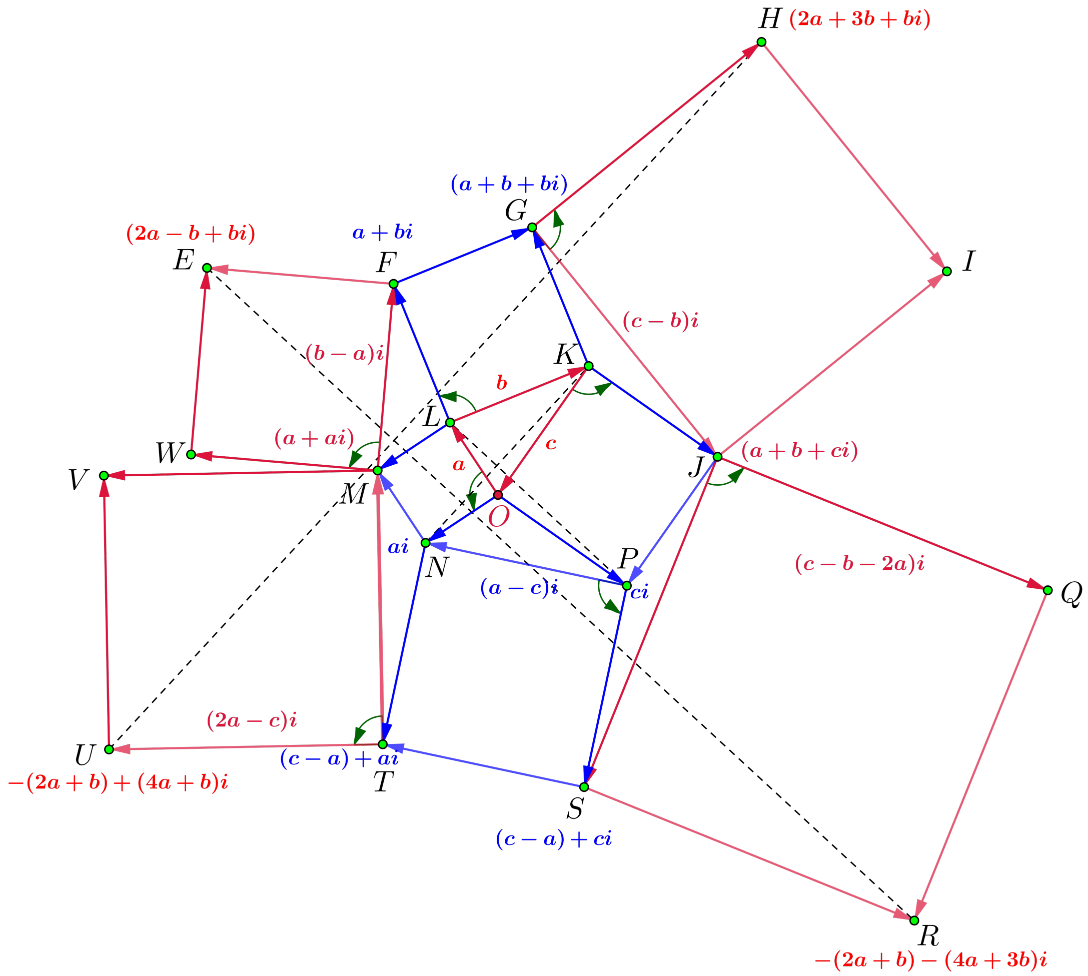


Figure 3.

Since $\overrightarrow{MF} = (\mathbf{b} - \mathbf{a})\mathbf{i}$ then $\overrightarrow{MW} = \overrightarrow{FE} = \mathbf{a} - \mathbf{b}$. Hence $E \equiv \mathbf{a} - \mathbf{b} + \mathbf{a} + \mathbf{bi} = (\mathbf{2a} - \mathbf{b}) + \mathbf{bi}$.

Since $\overrightarrow{TM} = (\mathbf{c} - \mathbf{2a})$ then $\overrightarrow{TU} = (\mathbf{2a} - \mathbf{c})\mathbf{i}$. Hence $U \equiv (\mathbf{c} - \mathbf{a}) + \mathbf{ai} + (\mathbf{2a} - \mathbf{c})\mathbf{i} = -(\mathbf{2a} + \mathbf{b}) + (\mathbf{4a} + \mathbf{b})\mathbf{i}$. $(\because \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0})$

Thus, we have computed complex representations of $H$, $U$, $E$, and $R$ as follows:

$$H \equiv (\mathbf{2a} + \mathbf{3b}) + \mathbf{bi}, \qquad U \equiv -(\mathbf{2a} + \mathbf{b}) + (\mathbf{4a} + \mathbf{b})\mathbf{i},$$
$$E \equiv (\mathbf{2a} - \mathbf{b}) + \mathbf{bi}, \qquad R \equiv -(\mathbf{2a} + \mathbf{b}) - (\mathbf{4a} + \mathbf{3b})\mathbf{i}.$$

If we continue this way we get the complex representations of $A$, $B$, $C$, and $D$ (see Figure 2) as follows:

$$A \equiv (\mathbf{4a} + \mathbf{8b}) + (\mathbf{a} + \mathbf{4b})\mathbf{i}, \qquad B \equiv -(\mathbf{7a} + \mathbf{3b}) + (\mathbf{12a} + \mathbf{4b})\mathbf{i},$$
$$C \equiv (\mathbf{4a} - \mathbf{4b}) + (-\mathbf{a} + \mathbf{3b})\mathbf{i}, \qquad D \equiv -(\mathbf{7a} + \mathbf{4b}) - (\mathbf{12a} + \mathbf{8b})\mathbf{i}.$$

Therefore,

$\overrightarrow{AB} = (-\mathbf{7a} - \mathbf{3b} + (\mathbf{12a} + \mathbf{4b})\mathbf{i}) - (\mathbf{4a} + \mathbf{8b} + (\mathbf{a} + \mathbf{4b})\mathbf{i}) = -(\mathbf{11a} + \mathbf{11b}) + \mathbf{11ai}$,

and

$\overrightarrow{CD} = (-\mathbf{7a} - \mathbf{4b} - (\mathbf{12a} + \mathbf{8b})\mathbf{i}) - (\mathbf{4a} - \mathbf{4b} + (-\mathbf{a} + \mathbf{3b})\mathbf{i}) = -\mathbf{11a} - (\mathbf{11a} + \mathbf{11b})\mathbf{i}$.

Clearly, $\overrightarrow{AB} = -i\overrightarrow{CD}$. Therefore, $AB \perp CD$ and $|AB| = |CD|$.

## 4. Another Property

Observe that

$$\overrightarrow{HU} = -(\mathbf{2a}+\mathbf{b}) + (\mathbf{4a}+\mathbf{b})\mathbf{i} - ((\mathbf{2a}+\mathbf{3b})+\mathbf{bi}) = -(\mathbf{4a}+\mathbf{4b}) + \mathbf{4ai}.$$

and

$$\overrightarrow{ER} = -(\mathbf{2a}+\mathbf{b}) - (\mathbf{4a}+\mathbf{3b})\mathbf{i} - ((\mathbf{2a}-\mathbf{b})+\mathbf{bi}) = -\mathbf{4a} - (\mathbf{4a}+\mathbf{4b})\mathbf{i}.$$

Clearly, $\overrightarrow{HU} = -i\overrightarrow{ER}$. Therefore, $HU \perp ER$ and $|HU| = |ER|$.

Again,

$$\overrightarrow{KN} = \mathbf{ai} - (\mathbf{a}+\mathbf{b}) = -(\mathbf{a}+\mathbf{b}) + \mathbf{ai}.$$

and

$$\overrightarrow{LP} = \mathbf{ci} - \mathbf{a} = -(\mathbf{a}+\mathbf{b})\mathbf{i} - \mathbf{a}.$$

Clearly, $\overrightarrow{KN} = -i\overrightarrow{LP}$. Therefore, $KN \perp LP$ and $|KN| = |LP|$.

Observe that

$$\overrightarrow{KN} = 4\overrightarrow{HU} = 11\overrightarrow{AB}.$$

If we take the ratio of successive lengths we get the sequence $\{f_1', f_3', f_5', \cdots\}$ where

$$f_1' = \frac{|KN|}{|HU|} = \frac{4}{1} = \frac{F_2+F_4}{F_0+F_2}, \quad f_2' = \frac{|HU|}{|AB|} = \frac{11}{4} = \frac{F_4+F_6}{F_2+F_4}, \cdots$$

and the pattern will be continued if we draw squares continuously outside of this configuration. Also, in each iteration, the perpendicularity and equality will hold. Observe that $f_1' = \frac{1}{f_1}$, $f_2' = \frac{1}{f_2}, \cdots$ where the sequence $f_{2n-1}$ is previously defined and shown converges to $\frac{1}{\phi^2}$. Therefore, the sequence $f_{2n-1}'$ converges to $\phi^2$.

## References

[1] H. Okumura, *Problems 2023-1*, SJM, Volume 7, 2023, pp.9-12

[2] V. R. Shrimali, *Solution for Problems 2 and 3 of Problems 2023-1*, SJM, Volume 7, 2023, pp. 16-19.

[3] `https://en.wikipedia.org/wiki/Fibonacci_sequence`