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# Solution of Problem 2023-1-6

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**Abstract.** A solution of Problem 6 in [3] and some related results are given.

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## 1. Introduction

In this paper we prove a generalization of Problem 6 in [3] and also present some related results.

**Problem.** *For a triangle $ABC$, assume that there is a circle of radius $\rho_a$ touching $CA$ and $AB$ from inside of $ABC$ and the semicircle of diameter $BC$ externally. Similarly there is a circle of radius $\rho_b$ touching $AB$ and $BC$ from inside of $ABC$ and the semicircle of diameter $CA$ externally. There also is a circle of radius $\rho_c$ touching $BC$ and $CA$ from inside of $ABC$ and the semicircle of diameter $AB$ externally (see Figure 1). Then show that the inradius of the triangle $ABC$ equals*

$$\frac{1}{2}\left(\rho_a+\rho_b+\rho_c+\sqrt{\rho_a^2+\rho_b^2+\rho_c^2}\right).$$

## 2. Generalization

The problem is generalized as follows:

**Theorem 2.1.** *Let $ABC$ be a triangle and assume, without loss of generality, that angles with vertices at $B$ and $C$ are acute. We denote by $\omega_a$ the circle of radius $\rho_a$ touching the semicircle of diameter $BC$ constructed on the same side as the point $A$ externally if $\angle BAC < 90°$ otherwise internally, where $\omega_a$ touches the sides $CA$ and $AB$ if $\angle BAC < 90°$ otherwise it touches the lines $CA$ and $AB$*

*from the side opposite to the incircle of $ABC$. Similarly we define $\omega_b$, $\rho_b$, $\omega_c$, $\rho_c$. Then the inradius of $ABC$ equals*

$$\frac{1}{2}\left(\rho_a+\rho_b+\rho_c+\sqrt{\rho_a^2+\rho_b^2+\rho_c^2}\right) \text{ if } ABC \text{ is acute.} \tag{1}$$

$$\frac{1}{2}\left(-\rho_a+\rho_b+\rho_c+\sqrt{\rho_a^2+\rho_b^2+\rho_c^2}\right) \text{ if } \angle CAB \geq 90^\circ. \tag{2}$$

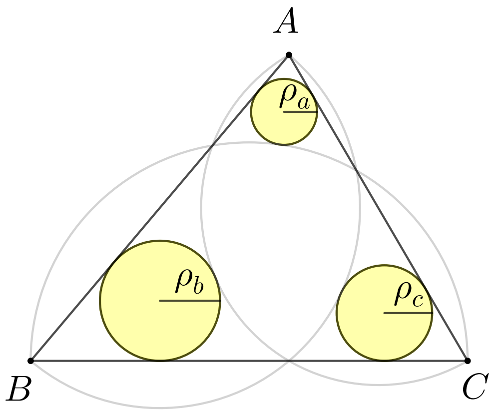


FIGURE 1.

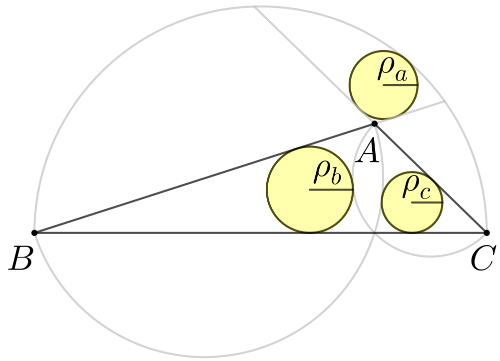


FIGURE 2.

## 3. PROOF OF THEOREM 2.1

For a triangle $ABC$, let $a$, $b$, $c$, $R$, $r$, $p$, $\Delta$, $I$, be the lengths of $BC$, $CA$, $AB$, the circumradius, the inradius, the semiperimeter, the area, the incenter, respectively. In the proof of Theorem 2.1 we will use the following lemmas.

**Lemma 1.** *The following identity holds:*

$$\left(b+c+\frac{ar}{p-a}\right)^2-2\left(b^2+c^2-a^2\right)=\left(a+\frac{(b+c)r}{p-a}\right)^2.$$

*Proof.* Using Heron's formula $\Delta=\sqrt{p(p-a)(p-b)(p-c)}$, and the well known identity $r=\frac{\Delta}{p}$, we have

$$\begin{aligned}
&\left(b+c+\frac{ar}{p-a}\right)^2-\left(a+\frac{(b+c)r}{p-a}\right)^2\\
=&\left(b+c+a+\frac{r(a+b+c)}{p-a}\right)\left(b+c-a-\frac{r(b+c-a)}{p-a}\right)\\
=&\,4p(p-a)\left(1+\frac{r}{p-a}\right)\left(1-\frac{r}{p-a}\right)\\
=&\,4p(p-a)-\frac{4p}{p-a}\cdot\frac{(p-a)(p-b)(p-c)}{p}\\
=&\,(a+b+c)(b+c-a)-(a-b+c)(a+b-c)\\
=&\,(b+c)^2-a^2-a^2+(b-c)^2=2\left(b^2+c^2-a^2\right).
\end{aligned}$$

□

**Lemma 2.** *The radius $\rho_a$ of the circle $\omega_a$ defined in Theorem 2.1, is given by the formula*

$$\rho_a = \pm r\left(1-\tan\frac{A}{2}\right),$$

*where the + sign is taken if $\angle BAC \le 90°$ and the − sign is taken if $\angle BAC > 90°$. Similar formulas hold for the radii $\rho_b$ and $\rho_c$ defined in the same way.*

*Proof.* Let $M$ be the midpoint of $BC$; let $D$ be the center $\omega_a$; let $E$, $G$ be the orthogonal projections of $D$ on $AC$, $AB$, respectively; let $J$ be the touch point of the incircle with the side $AC$. Let us first consider the case $\angle BAC \le 90°$. If $\angle BAC = 90°$ the circle $\omega_a$ reduces to a point, therefore $\rho_a = 0$ and the formula is verified since $\tan\frac{A}{2} = 1$. Therefore assume that $\angle BAC < 90°$ (see Figure 3).

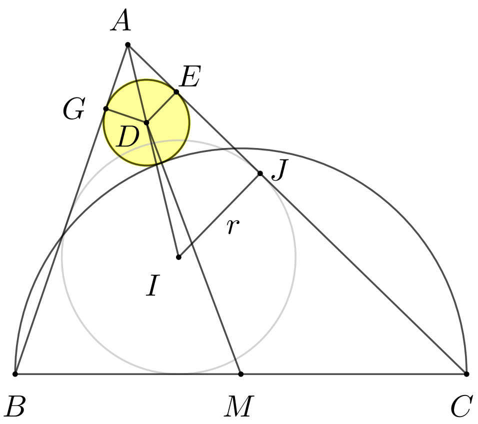


Figure 3.

Since $A$, $D$, $I$ are collinear we have $\angle GAD = \angle DAE = \frac{A}{2}$, so

$$AJ = r\cot\frac{A}{2} = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}}\cdot\sqrt{\frac{p(p-a)}{(p-b)(p-c)}} = p-a.$$

Denote $AE = AG = x$. From the similarity of the triangles $AJI$ and $AED$ we get

$$\frac{r}{p-a} = \frac{\rho_a}{x} \quad\Leftrightarrow\quad \rho_a = \frac{r}{p-a}\cdot x.$$

In the triangle $BDG$, since $\angle BGD = 90°$, we have

$$BD^2 = BG^2 + GD^2 = (c-x)^2 + \rho_a^2.$$

In the triangle $CED$, since $\angle CED = 90°$, we have

$$CD^2 = CE^2 + ED^2 = (b-x)^2 + \rho_a^2.$$

Since the circle $(D)$ and the semicircle of diameter $BC$ are externally tangent, we have $DM = \frac{a}{2} + \rho_a$. Then, using the median formula in triangle $BCD$ we get

$$4\cdot DM^2 = 2\cdot BD^2 + 2\cdot CD^2 - BC^2 \quad\Leftrightarrow$$

$$4\left(\frac{a}{2}+\rho_a\right)^2 = 2\left((c-x)^2+\rho_a^2\right) + 2\left((b-x)^2+\rho_a^2\right) - a^2 \quad\Leftrightarrow$$

(3) $$x^2 - \left(b+c+\frac{ar}{p-a}\right)x + \frac{b^2+c^2-a^2}{2} = 0.$$

Taking into account of Lemma 1, the discriminant of (3) equals to

$$\left(b+c+\frac{ar}{p-a}\right)^2 - 2\left(b^2+c^2-a^2\right) = \left(a+\frac{(b+c)r}{p-a}\right)^2.$$

Therefore the solutions of (3) are

$$x = p-a-r < p-a \qquad \text{and} \qquad x = p+\frac{pr}{p-a} > p-a.$$

Since $E \in AJ$ and $AJ = p-a$ we have $AE < AJ$, i.e. $x < p-a$. Hence we have

$$\rho_a = \frac{r}{p-a}\cdot x = \frac{r}{p-a}(p-a-r) = r\left(1-\frac{r}{p-a}\right) = r\left(1-\tan\frac{A}{2}\right).$$

The other case $\angle BAC > 90°$ can be proved similarly, taking into account that $DM = \frac{a}{2} - \rho_a$ because the circle $(D)$ and the semicircle of diameter $BC$ are internally tangent (see Figure 4).

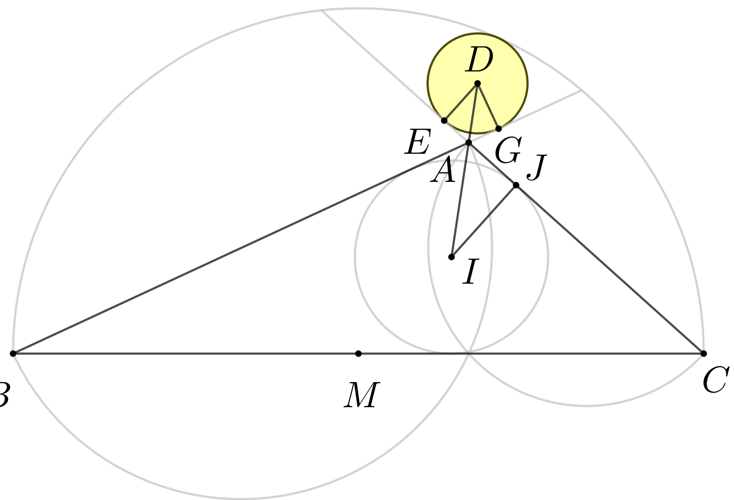


FIGURE 4.

□

**Lemma 3.** *The numbers* $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ *are the roots of the cubic*

$$px^3 - (4R+r)x^2 + px - r = 0.$$

*Proof.* The numbers $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ verify the equation

$$\left(x-\tan\frac{A}{2}\right)\left(x-\tan\frac{B}{2}\right)\left(x-\tan\frac{C}{2}\right) = 0 \quad \Leftrightarrow$$

$$x^3 - \left(\sum\tan\frac{A}{2}\right)x^2 + \left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)x - \prod\tan\frac{A}{2} = 0. \tag{4}$$

Now, using the well known identities[2]

$$\sum\tan\frac{A}{2} = \frac{4R+r}{p}, \qquad \sum\tan\frac{A}{2}\tan\frac{B}{2} = 1, \qquad \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{r}{p},$$

[2]See [2] pag.27, [1] pag. 358, [6] pag. 234, 237

the equation (4) rewrites as

$$x^3 - \frac{4R+r}{p} \cdot x^2 + x - \frac{r}{p} = 0 \quad \Leftrightarrow$$

$$px^3 - (4R+r)x^2 + px - r = 0. \tag{5}$$

□

**Proof of Theorem 2.1**. Let us first consider the case $\angle BAC \leq 90°$.
From Lemma 2 we get $\rho_a = r\left(1 - \tan\frac{A}{2}\right)$, hence $\tan\frac{A}{2} = 1 - \frac{\rho_a}{r}$.
Thus, using Lemma 3 we have that

$$p\left(1 - \frac{\rho_a}{r}\right)^3 - (4R+r)\left(1 - \frac{\rho_a}{r}\right)^2 + p\left(1 - \frac{\rho_a}{r}\right) - r = 0,$$

from which we get

$$p(r-\rho_a)^3 - r(4R+r)(r-\rho_a)^2 + pr^2(r-\rho_a) - r^4 = 0,$$
$$p\rho_a^3 + r(4R+r-3p)\rho_a^2 + 2r^2(2p-4R-r)\rho_a + 2r^3(2R+r-p) = 0.$$

Therefore $\rho_a$ and similarly $\rho_b$, $\rho_c$ satisfy the equation

$$px^3 + r(4R+r-3p)x^2 + 2r^2(2p-4R-r)x + 2r^3(2R+r-p) = 0.$$

Thus, using the Vieta's formulas we obtain

$$\rho_a + \rho_b + \rho_c = \frac{r(3p-4R-r)}{p}, \tag{6}$$

and

$$\sum \rho_a\rho_b = \frac{2r^2(2p-4R-r)}{p}, \tag{7}$$

from which it follows that

$$\begin{aligned} \rho_a^2 + \rho_b^2 + \rho_c^2 &= (\rho_a+\rho_b+\rho_c)^2 - 2\sum \rho_a\rho_b \\ &= \frac{r^2(3p-4R-r)^2}{p^2} - 2 \cdot \frac{2r^2(2p-4R-r)}{p} \\ &= \frac{r^2(p-4R-r)^2}{p^2}. \end{aligned} \tag{8}$$

Finally, using (6), (7) and (8) and taking into account the inequality[3] $4R+r > p$, we have

$$\rho_a + \rho_b + \rho_c + \sqrt{\rho_a^2 + \rho_b^2 + \rho_c^2} = \frac{r(3p-4R-r)}{p} + \frac{r(4R+r-p)}{p} = 2r.$$

so the formula (1) is proved.
Let us now consider the case where $\angle BAC > 90°$.
From Lemma 2 we get $\rho_a = r\left(\tan\frac{A}{2} - 1\right)$, hence $\tan\frac{A}{2} = 1 + \frac{\rho_a}{r}$. Thus, with a reasoning similar to that used in case $\angle BAC < 90°$, it can be shown that $-\rho_a$, $\rho_b$ and $\rho_c$ verify the equation

$$px^3 + r(4R+r-3p)x^2 + 2r^2(2p-4R-r)x + 2r^3(2R+r-p) = 0. \tag{9}$$

[3] See [2], pag. 49

Therefore by using the Vieta's formulas we obtain

$$-\rho_a+\rho_b+\rho_c=\frac{r(3p-4R-r)}{p}, \tag{10}$$

$$\rho_b\rho_c-\rho_a\rho_b-\rho_a\rho_c=\frac{2r^2(2p-4R-r)}{p}, \tag{11}$$

$$\rho_a^2+\rho_b^2+\rho_c^2=\frac{r^2(p-4R-r)^2}{p^2} \tag{12}$$

from which it follows that

$$-\rho_a+\rho_b+\rho_c+\sqrt{\rho_a^2+\rho_b^2+\rho_c^2}=\frac{r(3p-4R-r)}{p}+\frac{r(4R+r-p)}{p}=2r. \tag{13}$$

□

## 4. CONSTRUCTION OF CIRCLE $\omega_a$

The construction of the circle $\omega_a$ follows from the following theorem.

**Theorem 4.1.** *For a triangle $ABC$, let $\omega_a$ be the circle defined in Theorem 2.1. Let $I$ be the incenter of $ABC$ and let $J$, $E$ be the feet of the perpendiculars drawn on $AC$ from $I$ and $D$, respectively. We have $JE = JI$.*

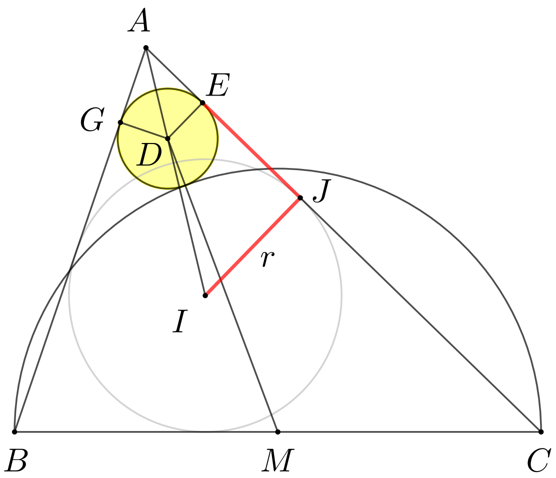


FIGURE 5.

*Proof.* If $\angle A < 90°$, from Lemma 2 we have $DE = r\left(1-\tan\frac{A}{2}\right)$. Therefore

$$\begin{aligned} JE = AJ - AE &= \frac{r}{\tan\frac{A}{2}} - \frac{DE}{\tan\frac{A}{2}} \\ &= \frac{r-DE}{\tan\frac{A}{2}} = \cot\frac{A}{2}\left(r-r+r\tan\frac{A}{2}\right) \\ &= r\cot\frac{A}{2}\tan\frac{A}{2} = r = JI. \end{aligned}$$

If $\angle A > 90°$ the proof is similar. □

The circle $\omega_a$ can be constructed in the following way (see figures 5 and 6)):

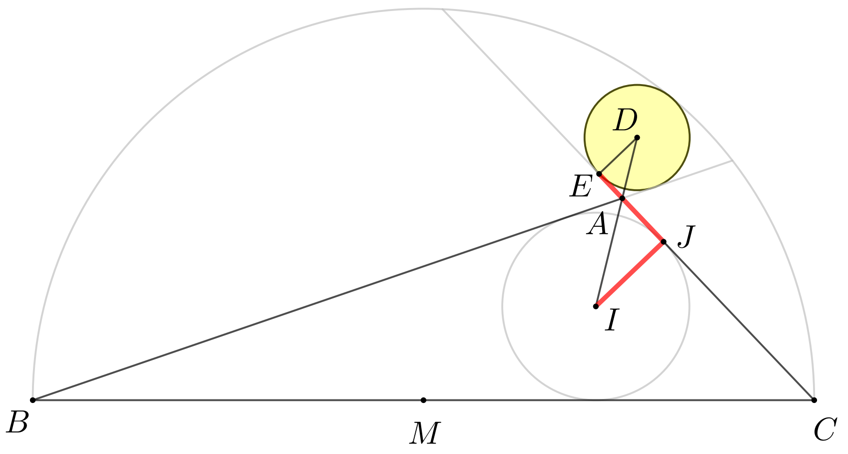


FIGURE 6.

- construct the incenter $I$ of $ABC$;
- construct the point $J$, orthogonal projection of $I$ on $AC$;
- construct the point $E \in AJ$ such that $JE = JI$;
- let $D$ be the intersection point of $AI$ with the perpendicular to $AC$ at $E$;
- draw the circle $\omega_a$ with center $D$ and radius $DE$.

The following corollary follows directly from theorem 4.1.

**Corollary 4.1.** *Let $ABC$ be a triangle, let $D$, $E$, $F$ be the centers of $\omega_a$, $\omega_b$, $\omega_c$ respectively; let $E_a$, $F_a$ be the feet of the perpendiculars drawn on $BC$ from $E$ and $F$, respectively. Define $D_b$, $F_b$ and $D_c$, $E_c$ cyclically. Then the six points $E_a$, $F_a$, $D_b$, $F_b$, $D_c$, $E_c$ lie on a circle with center $I$ and radius $\sqrt{2}r$. Furthermore we have $E_aF_a = D_bF_b = D_cE_c$ (see figures 7 and 8).*

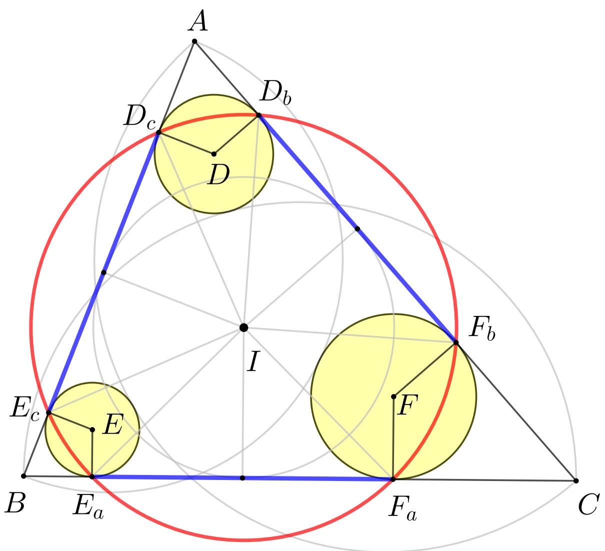


FIGURE 7.

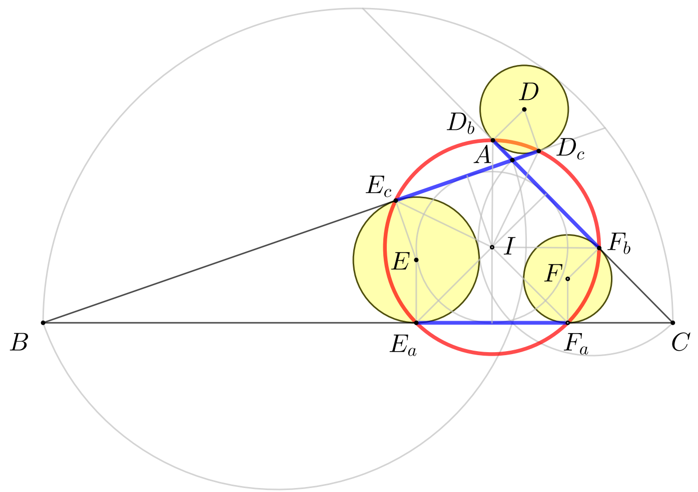


FIGURE 8.

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