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Solution for Problems 2 and 3 of Problems 2023-1

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Abstract. Using Pythogoras theorem and cosine formula for triangles, we solve the problems 2 and 3 in [1].

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1. Introduction

In this work, we provide the solutions for the Problems 2 and 3 in [1], which are stated as follows.

Problem 2. The followings are squares, where the vertices lie counterclockwise in these orders: $ABCD, DEFG, FCNH, GHIJ, INOK, JKLM, OPQL$. The point E lies on the segment CD, $a = |AB|, b = |DE|, c = |PQ|$ (see Figure 1). Show that

$$c = \sqrt{(5(a-b))^2 + (8b)^2}.$$

Problem 3. The followings are squares, where vertices lie counterclockwise in these orders: $ABCD, BEFG, JDHI, CGKH, IKLM, JMON, OLPQ$. (see Figure 2). Say something interesting for this figure.

2. Solution for Problem 2

We use the next proposition stated in [2].

Proposition 2.1. *The relation* $2(a^2+b^2) = c^2 + d^2$ *holds for the figure below.*

Referring to the Figure 1, since the point E lies on CD, the $\triangle CEF$ is a right angled triangle. Thus, by Pythogoras Theorem, we get $CF^2 = CE^2 + EF^2$. Therefore we have

$$CF^2 = (a-b)^2 + b^2. \tag{1}$$

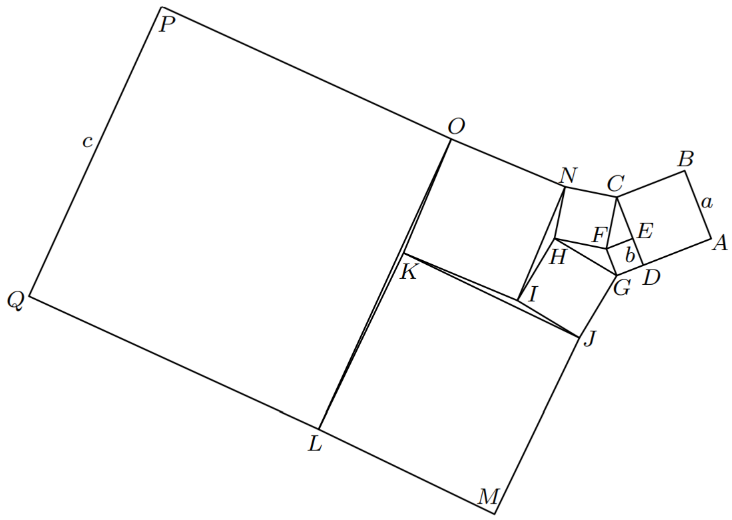

Figure 1.

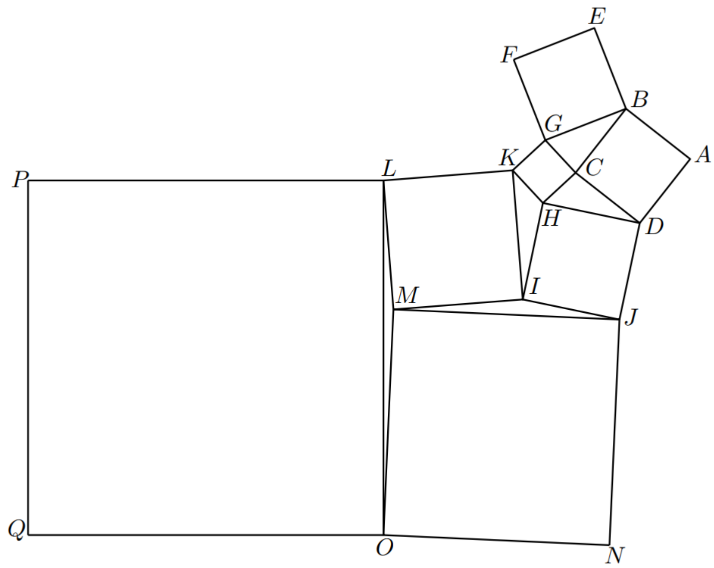

Figure 2.

Now, $\angle GFI = 2\pi - \angle CFI - \angle EFG - \angle CFE = \pi - \angle CFE$. This means that, $\cos\angle GFI = \cos(\pi - \angle CFE) = -\cos\angle CFE = -\frac{EF}{CF}$. Now in $\triangle GFI$ using cosine formula for triangle, we get:

$$GI^2 = GF^2 + FI^2 - 2(GF)(FI)\cos\angle GFI$$
$$\implies GI^2 = GF^2 + FI^2 + 2(GF)(FI)\frac{EF}{CF}$$
$$\implies GI^2 = GF^2 + FI^2 + 2(GF)^2$$
$$\implies GI^2 = 3(GF)^2 + FI^2.$$

By simplifying using equation (1), we get:

$$GI^2 = (a-b)^2 + (2b)^2\,. \tag{2}$$

Now we can represent HJ^2 using Proposition 2.1 as follows.

$$HJ^2 = 2\left(GI^2 + CF^2\right) - GF^2.$$

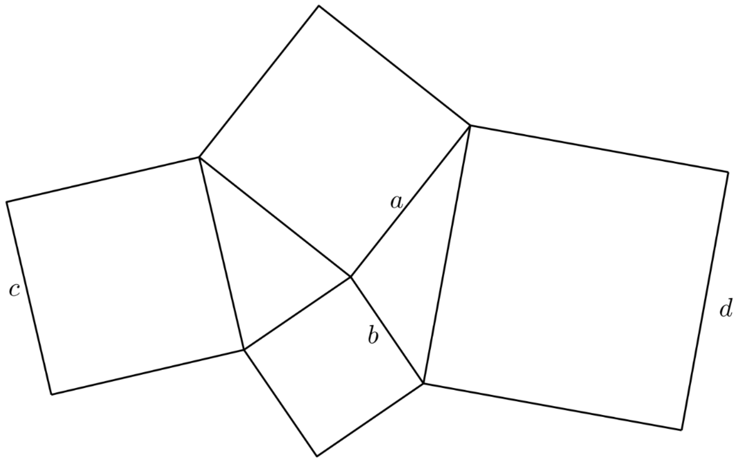

FIGURE 3.

Simplifying using equations (1) and (2), we get:

$$HJ^2 = (2\,(a-b))^2 + (3b)^2\,. \tag{3}$$

Following similar process for further squares, we get the following relations:

$$KM^2 = (3\,(a-b))^2 + (5b)^2\,, \tag{4}$$

$$LN^2 = (5\,(a-b))^2 + (8b)^2\,. \tag{5}$$

Equation (5) can be re-written as the following since $NLPQ$ is a square:

$$c = \sqrt{(5\,(a-b))^2 + (8b)^2}. \tag{6}$$

One interesting observation from the solution above is:

$$a_n^2 = (F_{n-2}\,(a-b))^2 + (F_{n-1}b)^2\,. \tag{7}$$

Where, $n \geq 2$, F_n refers to n^{th} Fibonacci number such that $F_0 = F_1 = 1$ and a_n refers to the side of the n^{th} square (in decreasing order of side length). Referring to Figure 1, this means $a_0 = a, a_1 = b, a_2 = CF, a_3 = GI, a_4 = HJ, a_5 = KM, a_6 = c$.

3. SOLUTION FOR PROBLEM 3

In the Figure 4, consider that $|AD| = a_0$, $|BE| = a_1$ and $\angle CBG = \alpha$.
By applying cosine formula for $\triangle GBC$ and assuming $|GC| = a_2$, we get:

$$GC^2 = GB^2 + BC^2 - 2\,(GB)\,(BC)\cos\alpha$$

$$\implies a_2^2 = a_1^2 + a_0^2 - 2a_1a_0\cos\alpha. \tag{8}$$

Referring to Figure 2, assuming $|HD| = a_3$ and by using Proposition 2.1, we get:

$$a_3^2 = 2\left(a_0^2 + a_2^2\right) - a_1^2. \tag{9}$$

By using equation (8), this can be written as:

$$a_3^2 = 2\left(a_0^2 + a_1^2 + a_0^2 - 2a_0a_1\cos\alpha\right) - a_1^2$$

$$\implies a_3^2 = 4a_0^2 - 4a_0a_1\cos\alpha + a_1^2. \tag{10}$$

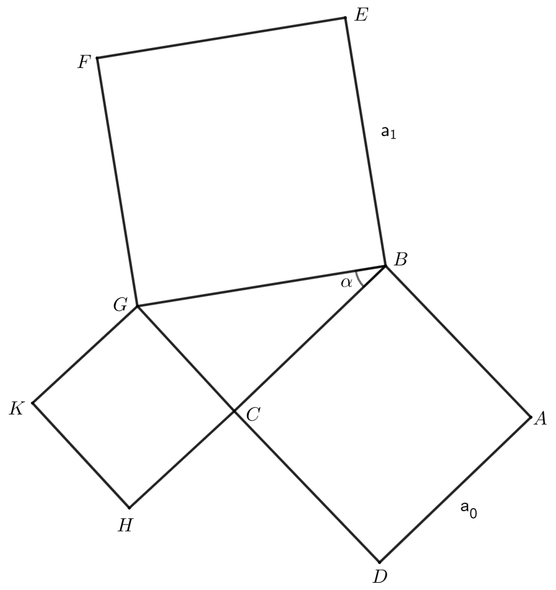

FIGURE 4.

Similarly, assuming $|IK| = a_4, |JM| = a_5$ and $|OL| = a_6$, and using Proposition 2.1 we get:

$$a_4^2 = 2\left(a_2^2 + a_3^2\right) - a_0^2, \tag{11}$$

$$a_5^2 = 2\left(a_4^2 + a_3^2\right) - a_2^2, \tag{12}$$

$$a_6^2 = 2\left(a_4^2 + a_5^2\right) - a_3^2. \tag{13}$$

Using the previous equations, these can be simplified into the following equations.

$$a_4^2 = 9a_0^2 + 4a_1^2 - 12a_1a_0\cos\alpha, \tag{14}$$

$$a_5^2 = 25a_0^2 + 9a_1^2 - 30a_0a_1\cos\alpha, \tag{15}$$

$$a_6^2 = 64a_0^2 + 25a_1^2 - 80a_0a_1\cos\alpha. \tag{16}$$

It can be observed from the solution that:

$$a_n^2 = (F_{n-1}a_0)^2 + (F_{n-2}a_1)^2 - 2\left(F_{n-1}a_0\right)\left(F_{n-2}a_1\right)\cos\alpha. \tag{17}$$

Where, $n \geq 2$ and F_n refers to n^{th} Fibonacci number such that $F_0 = F_1 = 1$. It is also interesting to note the difference between the figures from Problem 2 and Problem 3. The only difference in the construction comes from the fact that while for Problem 2, the point of the second square lies on the first square, that's no longer the case for Problem 3. The impact of this is observed in the additional $\cos\alpha$ term in equation (17). This in other words refers to the component of a_1 lying along a_0.

REFERENCES

[1] H. Okumura, Problems 2023-1, Sangaku J., Math., 9-12 (2023).
[2] Uchida ed., Kokonsankan, 1832, Tohoku University Digital Collection.