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Twin circles of a generalized arbelos with division by zero $1/0 = 0$

*Hiroshi Okumura and **Dao Thanh Oai
*Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com
**Cao Mai Doai, Quang Trung, Kien Xuong, Thai Binh, Viet Nam
e-mail: oaidt.evnpsc@gmail.com

Abstract. We give several pairs of congruent circles arising from a generalized arbelos. A special case, in which the congruent circles are lines, is also considered using division by zero $1/0 = 0$.

Keywords. congruent circles, generalized arbelos, division by zero

Mathematics Subject Classification (2010). 51M04

1. Introduction

For a point C on the segment AB, let α, β and γ be the semicircles of diameters BC, CA and AB, respectively constructed on the same side of AB. The area surrounded by the three semicircles is called the arbelos and the radical axis of α and β divides it into two curvilinear triangles with congruent incircles called *the twin circles of Archimedes* (see Figure 1). In this note we consider two pairs of semicircles of diameters CB_i and CA_i $(i = 1, 2)$ instead of the semicircles α and β such that the semicircles of diameter A_iB_i and γ are concentric (see Figure 2), and show the existence of several pairs of twin circles, i.e., two congruent circles.

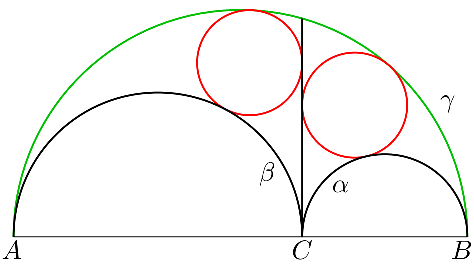

Figure 1.

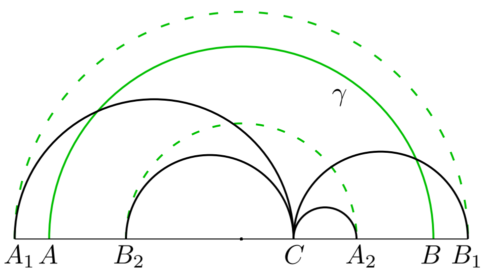

Figure 2.

If $A_1 = A_2$, then the configuration consists of three semicircles, which is called an arbelos with overhang ([1], [3]).

2. RESULTS

For points P and Q on the line AB, we denote the semicircle of diameter PQ constructed on the same side of AB as γ by (PQ). Let $|CB| = 2a$ and $|CA| = 2b$. We use a rectangular coordinate system with origin C such that B has coordinates $(2a, 0)$ and the farthest point on γ from AB has coordinates $(a-b, a+b)$.

If a circle or a semicircle touches one of given two circles internally and the other externally, we say that it touches the two circles in the opposite sense, otherwise in the same sense. Let each of α_1, β_1, α_2 and β_2 be a circle or a semicircle. If α_i touches β_i externally for $i = 1, 2$ or α_i touches β_i internally for $i = 1, 2$, then we say that α_1 touches β_1 and α_2 touches β_2 in the same sense, otherwise in the opposite sense.

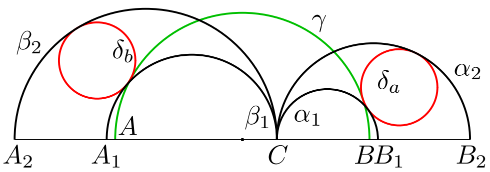

Figure 3: $(1, 1; 1, -1; 1)$.

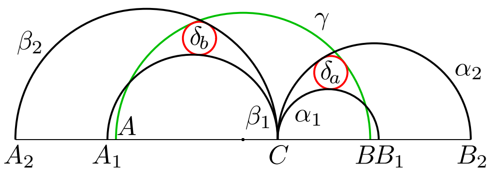

Figure 4: $(1, 1; 1, -1; -1)$.

Theorem 1. *For three concentric semicircles $\gamma = (AB)$, (A_1B_1) and (A_2B_2), let $\alpha_i = (CB_i)$ and $\beta_i = (CA_i)$. If there are two circles δ_a and δ_b such that they do not touch and γ touches δ_a and δ_b in the same sense, and δ_a touches α_i and δ_b touches β_i in the same sense for $i = 1, 2$, then δ_a and δ_b are congruent.*

Proof. Let the semicircle α_i (resp. β_i) have radius a_i (resp. b_i) and center of coordinates $(\sigma_i a_i, 0)$ (resp. $(-\sigma_i b_i, 0)$), where $\sigma_i = 1$ if $\overrightarrow{AB}$ and $\overrightarrow{A_iB_i}$ have the same direction, otherwise $\sigma_i = -1$ for $i = 1, 2$. We assume that the two circles δ_a and δ_b exist, and let $r_a > 0$ and (x_a, y_a) be the radius and the coordinates of the center of δ_a. Similarly $r_b > 0$ and (x_b, y_b) are defined. Considering the distances from the center of δ_a to the centers of α_i and γ for $i = 1, 2$, we have

$$(1) \qquad (x_a - \sigma_i a_i)^2 + y_a^2 = (\varsigma_i r_a + a_i)^2, \quad (x_a - (a-b))^2 + y_a^2 = (\tau r_a + a + b)^2,$$

where $\varsigma_i = 1$ if δ_a touches α_i externally, otherwise -1, and $\tau = 1$ if δ_a touches γ externally, otherwise -1. Similarly for $i = 1, 2$, we have

$$(2) \qquad (x_b + \sigma_i b_i)^2 + y_b^2 = (\varsigma_i r_b + b_i)^2, \quad (x_b - (a-b))^2 + y_b^2 = (\tau r_b + a + b)^2.$$

Firstly we consider the sixteen cases $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, \pm 1; \pm 1, \pm 1; \pm 1)$.

(i) Assume $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, 1; 1, 1; 1)$. Eliminating x_a and y_a from (1), and solving the resulting equation for r_a, we have $r_a = -a < 0$. Therefore the circle δ_a does not exist.

(ii) Assume $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, 1; 1, 1; -1)$. Then we have $r_a = b$ from (1) and $r_b = a$ from (2). Hence δ_a and δ_b have diameters CA and CB, respectively, i.e., they touch, a contradiction. Therefore the circles δ_a and δ_b do not exist.

(iii) Assume $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, 1; 1, -1; 1)$ (see Figure 3). Eliminating x_a and y_a from (1), and solving the resulting equation for r_a, we have

$$r_a = \frac{ab(a_1 - a_2)}{aa_2 - a_1(a_2 + b)}. \tag{3}$$

Similarly from (2), we have

$$r_b = \frac{ab(b_1 - b_2)}{bb_2 - b_1(b_2 + a)}.$$

Since $\sigma_1 = \sigma_2 = 1$, we have $a_i - b_i = a - b$ for $i = 1, 2$. Substituting $a_1 = a - b + b_1$ and $a_2 = a - b + b_2$ in (3), we get

$$r_a = \frac{ab(a_1 - a_2)}{aa_2 - a_1(a_2 + b)} = \frac{ab(b_1 - b_2)}{bb_2 - b_1(b_2 + a)} = r_b.$$

(iv) Assume $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, 1; 1, -1; -1)$ (see Figure 4). In a similar way as in (iii), we get

$$r_a = \frac{ab(a_1 - a_2)}{aa_1 - a_2(a_1 + b)} = \frac{ab(b_1 - b_2)}{bb_1 - b_2(b_1 + a)} = r_b.$$

(v) Assume $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, 1; -1, 1; 1)$. The result is obtained from (iii) by exchanging the suffixes $_1$ and $_2$ (see Figure 3), i.e.,

$$r_a = \frac{ab(a_2 - a_1)}{aa_1 - a_2(a_1 + b)} = \frac{ab(b_2 - b_1)}{bb_1 - b_2(b_1 + a)} = r_b.$$

(vi) Assume $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, 1; -1, 1; -1)$. The result is obtained from (iv) by exchanging the suffixes $_1$ and $_2$, i.e.,

$$r_a = \frac{ab(a_2 - a_1)}{aa_2 - a_1(a_2 + b)} = \frac{ab(b_2 - b_1)}{bb_2 - b_1(b_2 + a)} = r_b.$$

(vii) If $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, 1; -1, -1; 1)$, then we have $r_a = -b$. Therefore the circle δ_a does not exist.

(viii) If $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2, \tau) = (1, 1; -1, -1; -1)$, then the circles δ_a and δ_b do not exist similar as to the case (ii).

(ix) Assume $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, -1; 1, 1; 1)$ (see Figure 5). Then we have

$$r_a = \frac{-ab(a_1 + a_2)}{aa_2 - a_1(a_2 - b)} = \frac{-ab(b_1 + b_2)}{ab_1 - b_2(b_1 - b)} = r_b.$$

(x) Assume $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, -1; 1, 1; -1)$ (see Figure 6). Then we have

$$r_a = \frac{ab(a_1 + a_2)}{aa_1 + a_2(a_1 + b)} = \frac{ab(b_1 + b_2)}{bb_1 + b_2(b_1 + a)} = r_b.$$

(xi) If $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, -1; 1, -1; 1)$, then we have $r_a = -a$. Therefore the circle δ_a does not exist.

(xii) If $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, -1; 1, -1; -1)$, then we have $r_a = b$ and $r_b = a$. Hence the circles δ_a and δ_b do not exist similar as to the case (ii).

(xiii) If $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, -1; -1, 1; 1)$, then we have $r_a = -b$. Therefore the circle δ_a does not exist.

(xiv) If $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, -1; -1, 1; -1)$, then we have $r_a = a$ and $r_b = b$. Therefore the circles δ_a and δ_b do not exist similar as to the case (ii).

(xv) If $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, -1; -1, -1; 1)$, then by (1) we have

$$r_a = -\frac{ab(a_1 + a_2)}{aa_1 + a_2(a_1 + b)} < 0.$$

Hence the circle δ_a does not exist.

(xvi) Assume $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, -1; -1, -1; -1)$ (see Figure 7). Then we have

$$r_a = \frac{ab(a_1 + a_2)}{aa_2 - a_1(a_2 - b)} = \frac{ab(b_1 + b_2)}{ab_1 - b_2(b_1 - b)} = r_b.$$

From the above results, we can get the two congruent circles δ_a and δ_b in the seven cases $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, 1; 1, -1; 1)$, $(1, 1; 1, -1; -1)$, $(1, 1; -1, 1; 1)$, $(1, 1; -1, 1; -1)$, $(1, -1; 1, 1; 1)$, $(1, -1; 1, 1; -1)$, $(1, -1; -1, -1; -1)$.

There are still more sixteen cases $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (-1, \pm 1; \pm 1, \pm 1; \pm 1)$ to be considered. However the radii of δ_a and δ_b in the eight cases $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (-1, 1; \pm 1, \pm 1; \pm 1)$ are obtained from the already considered cases $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, -1; \pm 1, \pm 1; \pm 1)$ by exchanging the suffixes $_1$ and $_2$. Also the radii of δ_a and δ_b in the remaining cases $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (-1, -1; \pm 1, \pm 1; \pm 1)$ are obtained from the already considered cases $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (1, 1; \pm 1, \pm 1; \pm 1)$ by exchanging a_1 and b_1 and exchanging a_2 and b_2. Therefore we can get the two congruent circles δ_a and δ_b in the fourteen cases $(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2; \tau) = (\pm 1, 1; 1, -1; 1)$, $(\pm 1, 1; 1, -1; -1)$, $(\pm 1, 1; -1, 1; 1)$, $(\pm 1, 1; -1, 1; -1)$, $(\pm 1, -1; 1, 1; 1)$, $(\pm 1, -1; 1, 1; -1)$, $(\pm 1, -1; -1, -1; -1)$. □

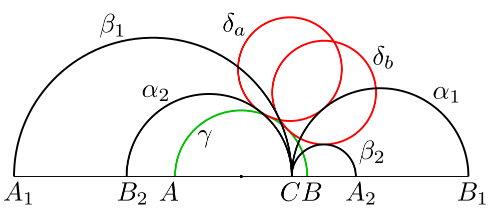

Figure 5: $(1, -1; 1, 1; 1)$.

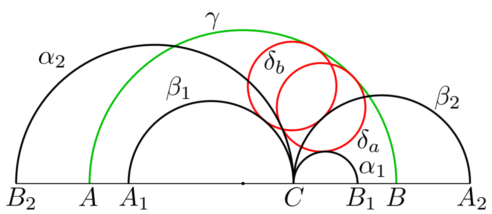

Figure 6: $(1, -1; 1, 1; -1)$.

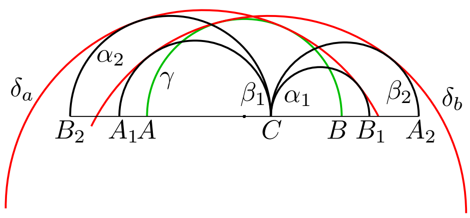

Figure 7: $(1, -1; -1, -1; -1)$.

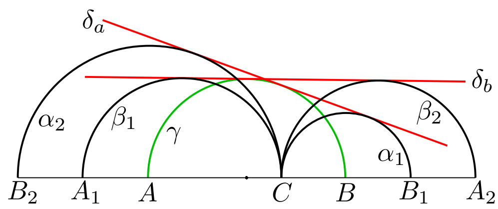

Figure 8: $r_a = r_b = 0$.

If $(\sigma_1, \sigma_2) = (1, -1)$, then we have $aa_2 - a_1(a_2 - b) = ab_1 - b_2(b_1 - b)$ by $a_1 = a - b + b_1$ and $a_2 = b - a + b_2$. Moreover if

$$aa_2 - a_1(a_2 - b) = ab_1 - b_2(b_1 - b) = 0 \tag{4}$$

in the cases (ix) and (xvi), then we have $r_a = r_b = 0$ by division by zero $1/0 = 0$ ([4]). Notice that a line has radius 0 as a circle ([2], [4]). Solving (4) for a and b, we have

$$(a, b) = \left(\frac{a_1 b_2 (b_1 - a_2)}{a_1 b_1 - a_2 b_2}, \frac{a_2 b_1 (a_1 - b_2)}{a_1 b_1 - a_2 b_2} \right),$$

which enable us to get the semicircle γ from the semicircles α_1, β_1, α_2 and β_2 so that δ_a and δ_b are lines. Indeed in this case, the semicircles α_1, α_2 and γ have an external common tangent, and the semicircles β_1, β_2 and γ also have an external common tangent (see Figure 8).

Division by zero was founded by a professor emeritus at Gunma University Saburou Saitoh. For an extensive reference of this entirely new concept, see [4].

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