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A rare sangaku problem involving three congruent circles

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Abstract. We consider a configuration of two externally touching circles with their external common tangents, and several congruent circles touching it.

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1. Introduction

We consider two externally touching circles α and β of radii a and b $(a < b)$, respectively with the external common tangents t and u meeting in a point V, which is denoted by $\mathcal{J}$. If $b = na$ for a real number n, the configuration $\mathcal{J}$ is explicitly denoted by $\mathcal{J}(n)$. For three figures f_1, f_2, f_3, each of which is a line or a circle, $\Delta(f_1, f_2, f_3)$ denotes the curvilinear triangle surrounded by the three figures, and $\mathrm{I}(f_1, f_2, f_3)$ is its incircle. Let γ_0 and γ_1 be circles congruent to $\gamma_2 = \mathrm{I}(\alpha, \beta, t)$ lying inside of $\Delta(\alpha, t, u)$ and touching t such that γ_0 touches u and γ_1 touches α (see Figure 1). In this paper we consider the following problem for $\mathcal{J}$ with the three congruent circles proposed by Kubodera (久保寺正福) in 1817, which is rare and can only be found in [5] and quoted in [4, p.45] (see Figure 2).

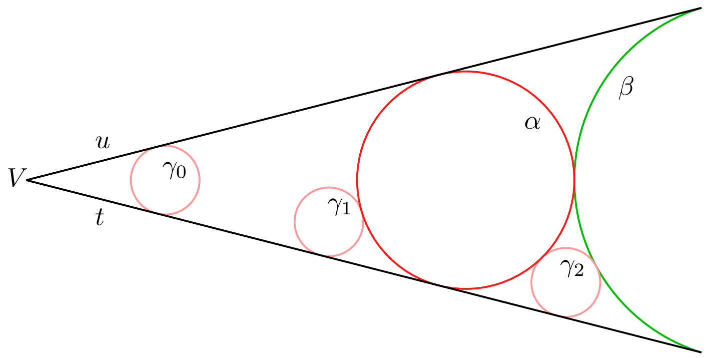

Figure 1: The configuration $\mathcal{J}$ with the congruent circles γ_0, γ_1 and γ_2.

Problem 1. If γ_0 touches γ_1, show $\mathcal{J} = \mathcal{J}(2)$.

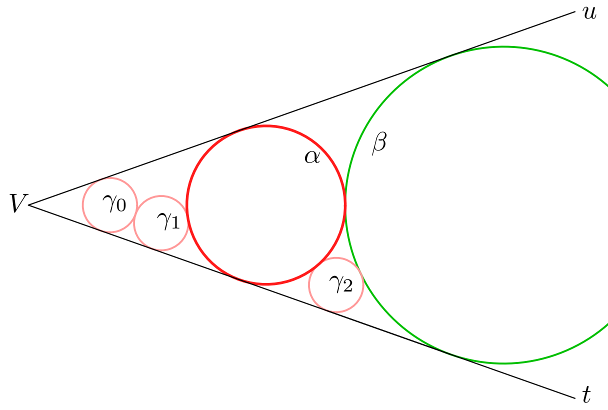

Figure 2: Kubodera's problem.

2. Generalization

A solution of Problem 1 can be found in [3]. In this section we generalize the problem. We use the next proposition.

Proposition 1. *If two externally touching circles C_1 and C_2 of radii r_1 and r_2 with an external common tangent s touch s at points P and Q, then*
(i) $|PQ| = 2\sqrt{r_1 r_2}$.
(ii) *If r_3 is the radius of* $\mathrm{I}(C_1, C_2, s)$, *then we have* $\dfrac{1}{\sqrt{r_3}} = \dfrac{1}{\sqrt{r_1}} + \dfrac{1}{\sqrt{r_2}}$.

Let A (resp. B, C) be the point of tangency of the circle α (resp. β, γ_0) and the line t for $\mathcal{J}$. Let v be the line joining V and the center of α. The following two triangles are similar (see Figure 3): (a) the triangle formed by the line v, the line parallel to t passing through the center of γ_0, and the perpendicular to t at A, (b) the triangle formed by v, the line parallel to t passing through the center of α, and the perpendicular to t at B. Let c be the radius of γ_0.

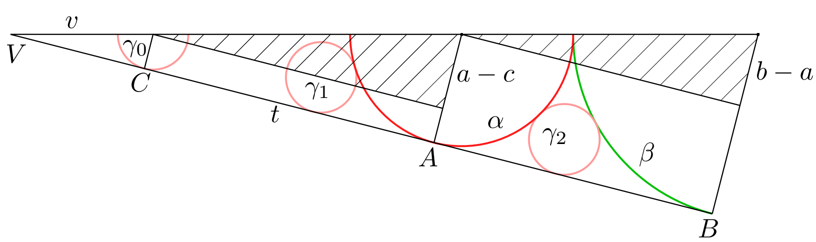

Figure 3: Similar triangles for Kubodera's problem.

The next theorem is a generalization of Problem 1.

Theorem 1. *The circles γ_0 and γ_1 touch if and only if $\mathcal{J} = \mathcal{J}(2)$.*

Proof. By Proposition 1(i), the circles γ_0 and γ_1 touch if and only if $|AC| = 2\sqrt{ac} + 2c$. Therefore by the similar triangles described in (a) and (b), γ_0 and γ_1 touch if and only if

$$\frac{2\sqrt{ac} + 2c}{a - c} = \frac{2\sqrt{ab}}{b - a}. \tag{1}$$

While by Proposition 1(ii), we get

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}. \tag{2}$$

This implies

$$\frac{\sqrt{c}}{\sqrt{a}-\sqrt{c}} = \frac{\sqrt{b}}{\sqrt{a}}.$$

Hence we have

$$\frac{\sqrt{ac}+c}{a-c} - \frac{\sqrt{ab}}{b-a} = \frac{\sqrt{c}}{\sqrt{a}-\sqrt{c}} - \frac{\sqrt{ab}}{b-a} = \frac{\sqrt{b}}{\sqrt{a}} - \frac{\sqrt{ab}}{b-a} = \frac{\sqrt{b}(b-2a)}{\sqrt{a}(b-a)}.$$

Therefore (1) holds if and only if $b = 2a$. □

In the event of the theorem, we have $c = 2(3-2\sqrt{2})a$ by (2).

3. Circles of radius $2\left(3+\sqrt{2}\right)a$ for $\mathcal{J}(2)$

In this section we consider several circles of radius $2\left(3+\sqrt{2}\right)a$ for $\mathcal{J}(2)$. Let $\varepsilon_{-3}(\neq \gamma_2)$ be the circle touching α and β externally and t from the same side as α for $\mathcal{J}$. Let ε_{-1} and ε_0 be the circles congruent to ε_{-3} touching t from the same side as α such that ε_0 touches u from the same side as α, and $\varepsilon_{-1}(\neq \varepsilon_{-3})$ touches α externally. Let e be the radius of the circle ε_0.

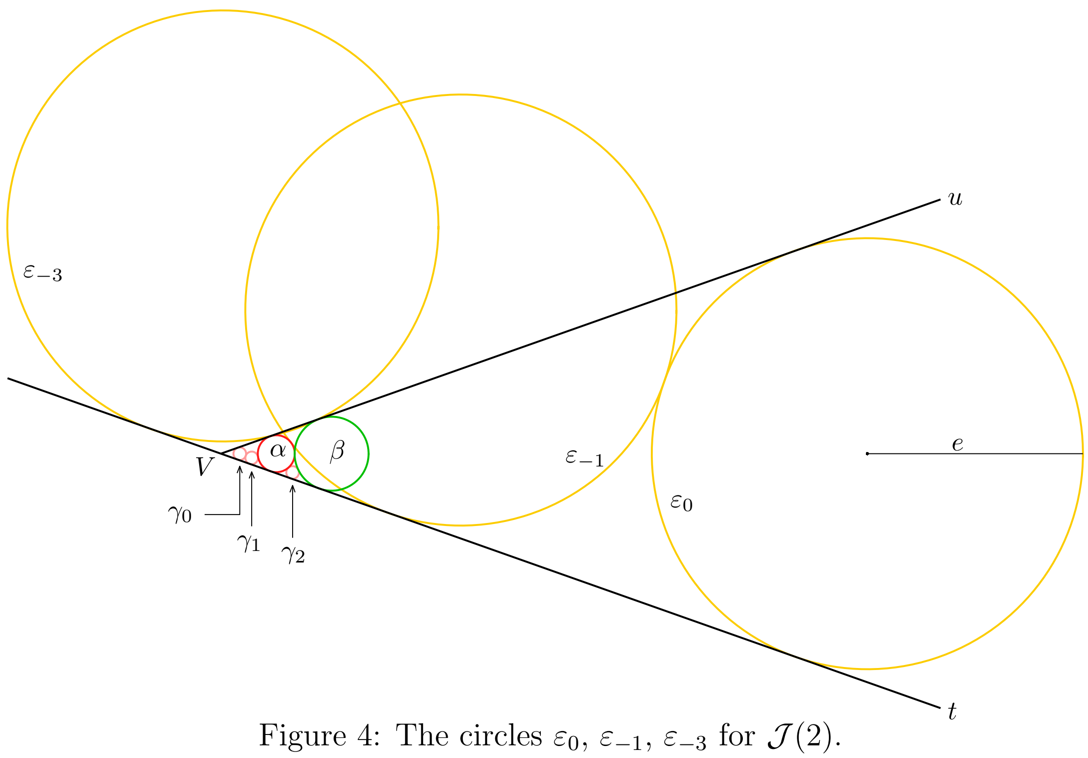

Figure 4: The circles ε_0, ε_{-1}, ε_{-3} for $\mathcal{J}(2)$.

Theorem 2. *The circles ε_0 and ε_{-1} touch if and only if $\mathcal{J} = \mathcal{J}(2)$.*

Proof. By the similar triangles made by v, the line parallel to t passing thorough the center of α and the perpendicular from the center of ε_0 to t and the triangle described in (b), the circles ε_0 and ε_1 touch if and only if the following relation holds:

$$\frac{2\sqrt{ae}+2e}{e-a}=\frac{2\sqrt{ab}}{b-a}. \tag{3}$$

The rest of the proof is similar to that of Theorem 1, and is omitted. □

In the event of the theorem, we get $e = 2(3+2\sqrt{2})a$. The circle ε_{-3} is an excircle of $\Delta(\alpha, \beta, t)$, while $\gamma_2 = \mathrm{I}(\alpha, \beta, t)$. Therefore we may say that *Theorem 2 is an excircle version of Theorem 1* (see Figure 4).

If $\mathcal{J} = \mathcal{J}(2)$, then the ratio of the sides of the two triangles in (a) and (b) equals $1 : 2\sqrt{2} : 3$. From now on we use a rectangular coordinate system with origin V so that the center of α has coordinates $(3a, 0)$, and the line t has an equation $x + 2\sqrt{2}y = 0$ for $\mathcal{J}(2)$. We get the next theorem (see Figure 5).

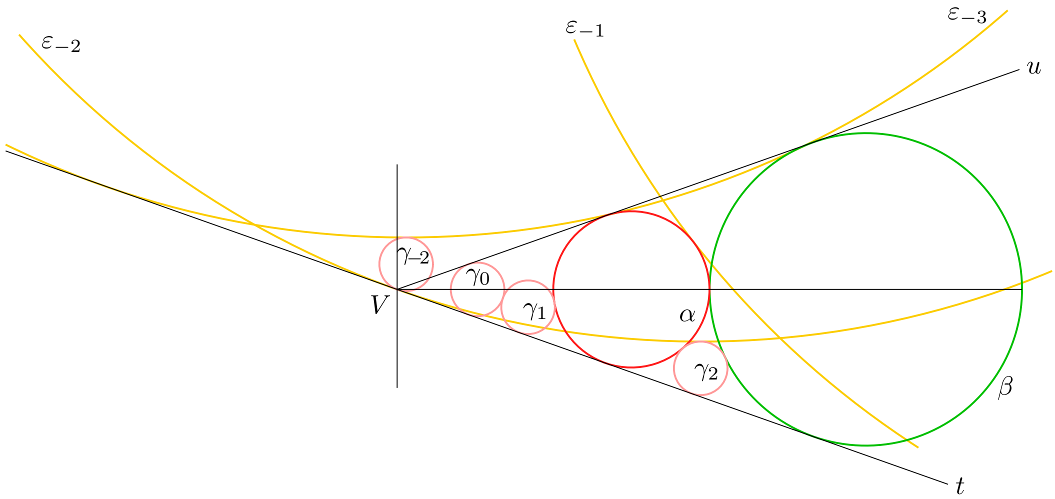

Figure 5: ε_{-3} touches γ_{-2} at the highest point on γ_{-2} for $\mathcal{J}(2)$.

Theorem 3. *The following statements hold for* $\mathcal{J}(2)$*.*
(i) *Let γ_{-2} be the circle touching the circle ε_{-3} externally at the lowest point on ε_{-3} and passing through the point V. Then γ_{-2} touches the line t at V and is congruent to the circle γ_0.*
(ii) *Let ε_{-2} be the circle touching the circle γ_2 externally at the highest point on γ_2 and passing through the point V. Then ε_{-2} touches t at V and is congruent to the circle ε_0.*

Proof. If (p, q) are the coordinates of the center of the circle ε_{-3}, then we have $(p-3a)^2 + q^2 = (e+a)^2$ and $(p-6a)^2 + q^2 = (e+b)^2$, since ε_{-3} touches α and β externally. Eliminating b and e and solving the resulting equations for p and q, we get

$$(p, q) = \left(\frac{2\left(3-2\sqrt{2}\right)}{3}a, \frac{4\left(5+3\sqrt{2}\right)}{3}a\right) = \left(\frac{c}{3}, \frac{4\left(5+3\sqrt{2}\right)}{3}a\right). \tag{4}$$

Let r be the radius of γ_{-2}. The center of γ_{-2} has coordinates $(p, q-e-r)$. Solving the equation $(0-p)^2 + (0-(q-e-r))^2 = r^2$ for r, we have $r = 2\left(3-2\sqrt{2}\right)a = c$. Hence γ_{-2} is congruent to γ_0 and has center of coordinates $(p, q-12a)$. While $(q-12a)/p = 2\sqrt{2}$. Therefore the line joining V and the center of γ_{-2} is perpendicular to t, i.e., γ_{-2} touches t at V. This proves (i). The part (ii) is proved similarly, where the center of γ_2 has coordinates

$$(x_2, y_2) = \left(\frac{2\left(3+2\sqrt{2}\right)}{3}a, \frac{4\left(-5+3\sqrt{2}\right)}{3}a \right) = \left(\frac{e}{3}, \frac{4\left(-5+3\sqrt{2}\right)}{3}a \right). \tag{5}$$

□

Notice that the coordinates of the centers of ε_{-3} and γ_2 differ only in signs.

4. THE CONFIGURATION $\mathcal{K}$

Let τ be the homothety of center V such that $\gamma_0^\tau = \beta$ for $\mathcal{J}(2)$. The homothety ratio equals $b/c = 3+2\sqrt{2}$, which is denoted by ρ. For a circle C, C^τ is denoted by C^1 and C^{τ^k} is denoted by C^k for an integer k, where $C^0 = C$. Then $(C^k)^j = C^{k+j}$ holds for an integer j. Let $\gamma_{-3} = \varepsilon_{-3}^{-2}$, and $\gamma_{-1} = \varepsilon_{-1}^{-2}$. Then γ_{-3} and γ_{-1} are congruent to γ_0. We now have the six congruent circles γ_k of radius c for $k = -3, -2, -1, 0, 1, 2$. We now define $\beta_k = \gamma_k^1$ and $\varepsilon_k = \gamma_k^2$. Then $\beta_0 = \beta$. Though we have already defined the congruent circles ε_{-3}, ε_{-2}, ε_{-1}, ε_0, our definition does not conflict with the former definitions. In this section we consider the following configuration $\mathcal{K}$ (see Figure 6).

$$\mathcal{K} = (\cup_{i\in\{-3,-2,-1,0,1,2\},j\in\mathbb{Z}}\gamma_i^j) \cup (\cup_{k\in\mathbb{Z}}\alpha^k) \cup \{t, u\}.$$

Before considering $\mathcal{K}$, we show three more circles congruent to γ_0 using τ.

Theorem 4 ([2]). *For two externally touching circles C_1 and C_2 of radii r_1 and r_2 with one of their externally common tangents s, let D_1 and D_2 be touching congruent circles of radius d lying inside of $\Delta(C_1, C_2, s)$ and touching s such that D_1 touches C_1 and D_2 touches C_2. Then we have*

$$d = \frac{w - \sqrt{w^2 - 4r_1r_2}}{2}, \quad \textit{where} \quad w = r_1 + r_2 + 4\sqrt{r_1r_2}.$$

If $r_1 = r_2 = r$ in the theorem, we have $d = (3-2\sqrt{2})r$. Let γ_3 and γ_4 be touching congruent circles lying inside of $\Delta(\beta_0, \beta_1, t)$ and touching t such that γ_3 touches β_0 and γ_4 touches β_1. Then γ_3 and γ_4 have radius $2\left(3-2\sqrt{2}\right)a = c$.

Let $\gamma_5 = \mathrm{I}(\varepsilon_{-1}, \alpha^1, t)$ and assume that γ_5 has center of coordinates (p, q) and radius r. Then we have $(p - \rho^2 x_{-1})^2 + (q - \rho^2 y_{-1})^2 = (r + \rho^2 c)^2$, $(p - \rho 3a)^2 + (q - 0)^2 = (r + \rho a)^2$, $(p + 2\sqrt{2}q)/3 = r$. Eliminating p and q from the three equations and solving the resulting equations for r, we get $r = c$ or $r = e$. Since $r < e$, we have $r = c$.

We now consider the configuration $\mathcal{K}$. Let (x_k, y_k) be the coordinates of the center of the circle γ_k. By the proof of Theorem 3, the circle γ_{-2} has center of coordinates

$$(x_{-2}, y_{-2}) = \left(\frac{c}{3}, \frac{4\left(-4+3\sqrt{2}\right)}{3}a \right). \tag{6}$$

The circle γ_0 has center of coordinates $(x_0, y_0) = (3c, 0)$. Solving the equations $(y_1 - y_0)/(x_1 - x_0) = -1/2\sqrt{2}$ and $(x_1 - x_0)^2 + (y_1 - y_0)^2 = (c + c)^2$ for x_1 and

y_1, we get

$$(x_1, y_1) = \left(\left(\frac{22}{3} - 4\sqrt{2}\right) a, \left(-4 + \frac{8\sqrt{2}}{3}\right) a\right). \tag{7}$$

Since the center of γ_0 is the midpoint of the segment joining the centers of γ_{-1} and γ_1, we get

$$(x_{-1}, y_{-1}) = \left(\left(\frac{86}{3} - 20\sqrt{2}\right) a, \left(4 - \frac{8\sqrt{2}}{3}\right) a\right). \tag{8}$$

The circle γ_2 has center of coordinates given by (5).

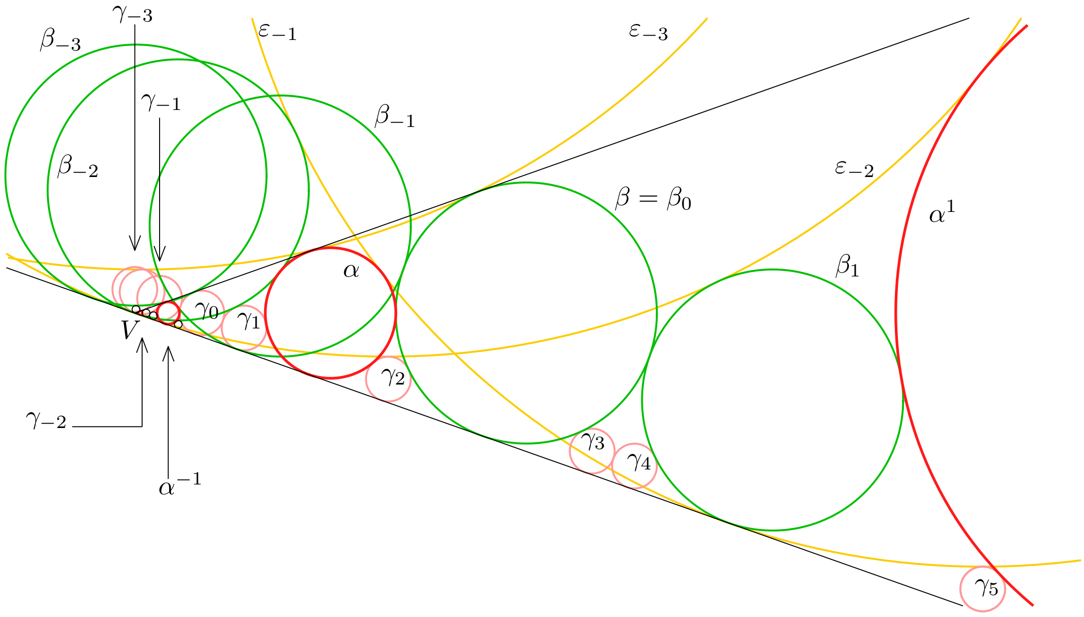

Figure 6: The configuration $\mathcal{K}$.

We consider relationships between the circles in $\mathcal{K}$.

The circles γ_{-3} and γ_{-2}, and the circles γ_{-2} and γ_2. Theorem 3 shows that γ_{-3}^{k+1} touches γ_{-2}^{k-1} externally at the highest point on γ_{-2}^{k-1}, and γ_{-2}^{k+1} touches γ_2^{k-1} externally at the highest point on γ_2^{k-1} for an integer k.

The circles γ_{-1} and γ_1 and t. Using (7) and solving the equations $(x - x_1)^2 + (y - y_1)^2 = c^2$ and $x + 2\sqrt{2}y = 0$ for x and y, we get the coordinates of the point of tangency of γ_1 and t, which are given by

$$\left(\frac{8(2-\sqrt{2})}{3} a, \frac{4\left(1-\sqrt{2}\right)}{3} a\right). \tag{9}$$

Using (8) and solving the equations $(x - \rho x_{-1})^2 + (y - \rho y_{-1})^2 = (\rho c)^2$ and $x + 2\sqrt{2}y = 0$ for x and y, we get the coordinates of the point of tangency of β_{-1} and t, which are also given by (9). Therefore the circles β_{-1}, γ_1 and the line t touch at a point, i.e., the circles γ_{-1}^{k+1}, γ_1^k and the line t touch at a point for an integer k, where γ_{-1}^{k+1}, γ_1^k touch internally.

The circles γ_{-3} and γ_0. The circles ε_{-3} and β_0 touch externally by the definition. Therefore the circles γ_{-3}^{k+1} and γ_0^k touch externally for an integer k.

The circles γ_{-2} and γ_{-1}. By (6) and (8), $(x_{-2}-\rho x_{-1})^2+(y_{-2}-\rho y_{-1})^2 = (c+\rho c)^2$ holds, i.e., γ_{-2} touches γ_{-1}^1 externally. Hence γ_{-2}^{k-1} and γ_{-1}^k touch externally for an integer k.

The circles γ_{-2} and γ_1. Since the equation $(\rho x_{-2}-x_1)^2+(\rho y_{-2}-y_1)^2 = (\rho c+c)^2$ holds by (6) and (7), the circles γ_{-2}^1 and γ_1 touch externally. Therefore the circles γ_{-2}^{k+1} and γ_1^k touch externally for an integer k.

The circle α. The circle α and γ_{-3}^2 touch externally by the definition. From $(\rho x_{-2}-3a)^2+(\rho y_{-2}-0)^2 = (\rho c+a)^2$, the circles α and γ_{-2}^1 touch externally. The circles α and γ_{-1}^2 touch externally by the definition of ε_{-1}. Therefore α touches the circles γ_{-3}^2, γ_{-2}^1, γ_{-1}^2, γ_1, γ_2 externally.

5. THE CONFIGURATION $\mathcal{J}(4)$

In this section we give a slight generalization of an unexpected result for $\mathcal{J}(4)$, which can be found in [3] and quoted in [1] (see Figure 7).

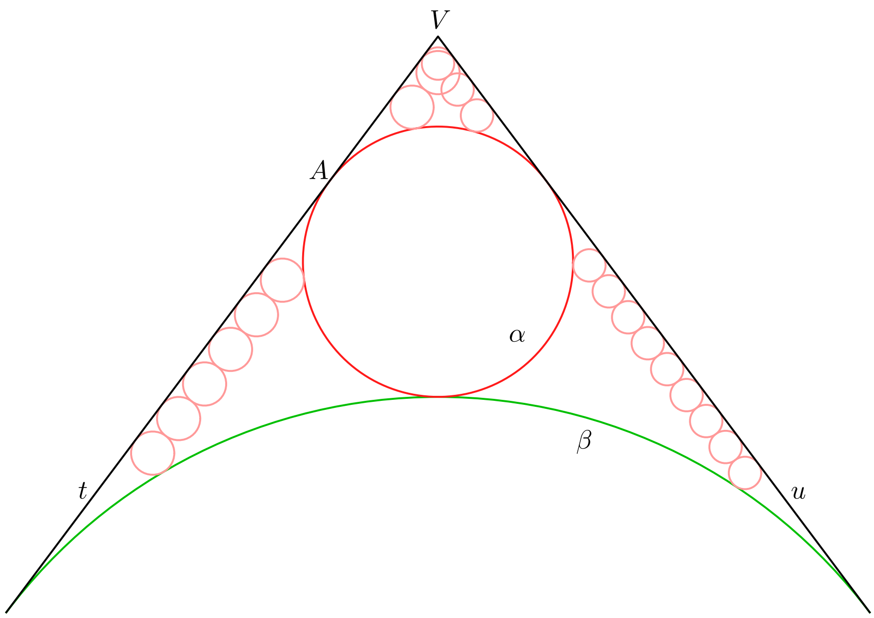

Figure 7: $(n, m) = (2, 6), (3, 9)$.

Let γ and γ' be congruent circles of radius c and centers C and C', respectively, touching t from the same side as α such that γ touches u from the same side as α and γ' touches α from the side opposite to β. Let $n = \sigma|CC'|/(2c)+1$, where $\sigma = 1$, if $\overrightarrow{CC'}$ and $\overrightarrow{VA}$ have the same direction, otherwise $\sigma = -1$. In this case we have $\sigma|CC'| = 2(n-1)c$ and say that *there are n congruent circles of radius c on t lying inside of $\Delta(\alpha, t, u)$*. If n is a positive integer, there actually exist circles $\gamma = \gamma_1$, γ_2, γ_3, $\cdots$, $\gamma_n = \gamma'$ of radius c such that they touch t from the same side, and γ_1 and γ_2 touch, γ_{i+1} touches γ_i at the farthest point on γ_i from γ_{i-1} for $i = 2, 3, 4, \cdots n-1$. Similarly let δ and δ' be congruent circles of radius c and centers D and D', respectively, touching t from the same side as α such that δ touches α from the same side as β and δ' touches β from the same side as α. Let $m = \sigma|DD'|/(2c)+1$, where $\sigma = 1$ if $\overrightarrow{DD'}$ and $\overrightarrow{VA}$ have the same direction otherwise $\sigma = -1$. In this case we have $\sigma|DD'| = 2(m-1)c$ and say that *there are m congruent circles of radius c on t lying inside of $\Delta(\alpha, \beta, t)$*.

Theorem 5. *Assume $b = 4a$. There are n congruent circles of radius c on t lying inside of $\Delta(\alpha, t, u)$ if and only if there are $3n$ congruent circles of radius c on t lying inside of $\Delta(\alpha, \beta, t)$.*

Proof. By the similar triangles described in (a) and (b) in section 2, there are n congruent circles of radius c on t lying inside of $\Delta(\alpha, t, u)$ if and only if

$$\frac{2\sqrt{ac} + 2(n-1)c}{a-c} = \frac{2\sqrt{ab}}{b-a}.$$

Similarly there are m congruent circles of radius c on t lying inside of $\Delta(\alpha, \beta, t)$ if and only if

$$2\sqrt{ac} + 2(m-1)c + 2\sqrt{bc} = 2\sqrt{ab}.$$

Substituting $b = 4a$ and $c = ja$ in the two equations, and solving the resulting equations for n and m, we get

$$n = \frac{(\sqrt{j}-1)(\sqrt{j}-2)}{3j}, \quad m = \frac{(\sqrt{j}-1)(\sqrt{j}-2)}{j}.$$

Therefore we get $m = 3n$. The theorem is proved. □

Notice that the theorem is true if n or m is not an integer (see Figure 8).

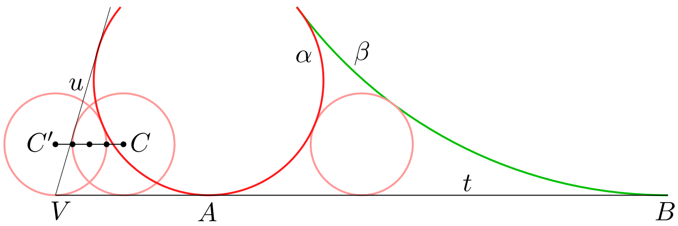

Figure 8: $(n, m) = (\frac{1}{3}, 1)$.

REFERENCES

[1] H. Okumura, Wasan Geometry, In: Sriraman B. (eds) Handbook of the Mathematics of the Arts and Sciences. 2021 Springer, Cham. https://doi.org/10.1007/978-3-319-70658-0_122-1

[2] H. Okumura, Theorems on two congruent circles on a line, Sangaku J. Math., **1** (2017) 35–38.

[3] H. Okumura, C. Sodeyama, A surprising property of successively touching circles, Mathematics Plus, **6** (1998) 17–18.

[4] Saitama Prefectural Library (埼玉県立図書館) ed., The Sangaku in Saitama (埼玉の算額), Saitama Prefectural Library (埼玉県立図書館), 1969.

[5] no author's name, Kururisha Sandaishū (久留里社算題集), no date, Tohoku University Digital Collection.