Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 6 (2022), pp. 23-25
Received 17 May 2022. Published on-line 19 May 2022
web: http://www.sangaku-journal.eu/

A note on circles touching two circles in a Pappus chain: Part 2

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. A result similar to the result for the circles touching two consecutive circles at their point of tangency in a Pappus chain in [2] is given.

Keywords. Pappus chain, orthogonal figures as touching figures, 1/0=0.

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

In [2] we have considered a chain of circles whose members touch two internally touching circles β and γ, and a circle touching two consecutive circles in the chain at their point of tangency. Then we have showm that a simple relationship between the radius of the circle and the radii of β and γ holds using division by zero $1/0 = 0$ [4]. In this note we consider a chain of circles whose members touch two externally touching circles, and show that a similar relationship is also true. The author considers that such a chain can also be called a Pappus chain.

2. Result

Let C be a point on a segment AB such that $|AC| = 2b$, $|BC| = 2a$ and $c = a + b$ $(a \neq b)$. The semicircles of diameters BC and AC constructed on the same side of AB are denoted by α and β, respectively. γ_1, γ_2, γ_3, $\cdots$ are the chain of circles touching α and β such that γ_1 touches the line AB (see Figures 1 and 2). If we invert the figure in the circle with center C orthogonal to γ_n, the images of γ_1, γ_2, γ_3, $\cdots$ are the circles congruent to γ_n and their centers lie on the perpendicular from the center of γ_n to AB. Therefore there are circles δ_1, δ_2, δ_3, $\cdots$ such that δ_i touches γ_i and γ_{i+1} at their point of tangency and AB at C, where we define δ_0 is the line AB. Let d_n be the radius of δ_n. We have the next theorem.

Theorem 1. *For a non-negative integer n, we have*

$$d_n = \frac{ab}{cn}.$$

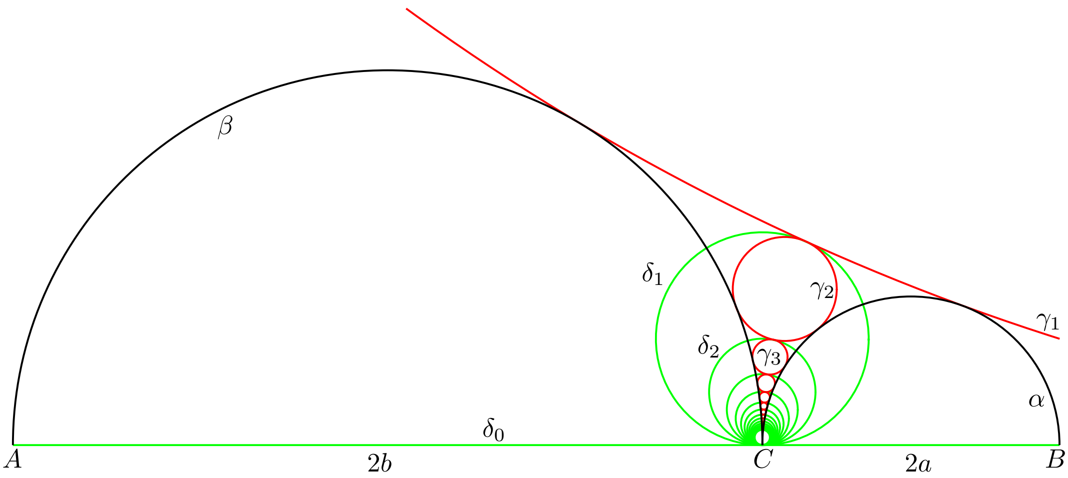

Figure 1.

If we use a rectangular coordinate system with origin C so that the farthest point on α has coordinates (a, a), the proof is similar to that of Theorem 1 in [2]. Therefore we omit the proof. Notice that the theorem is true in the case $n = 0$, since $1/0 = 0$ and $d_0 = 0$, because a line can be considered to be a circle of radius 0 as stated in [2, Section 2].

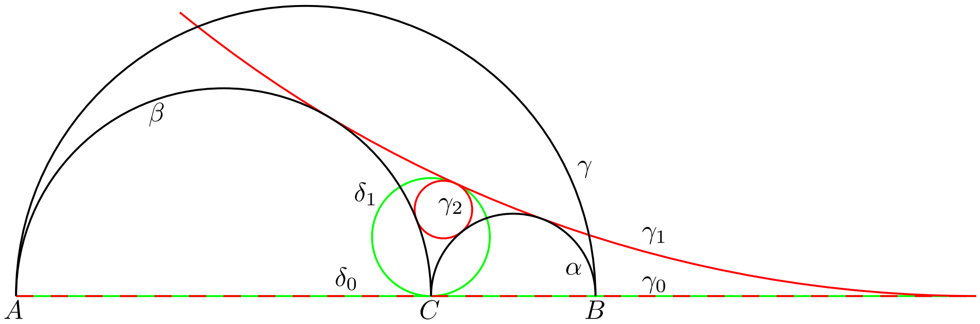

Figure 2.

Assume that two figures have a point P in common and the angle between the tangent lines at P equals θ. Then the two figures are said to touch at P if and only if $\tan\theta = 0$. While the angle between the tangent lines at the point of intersection equals $\frac{\pi}{2}$ for two orthogonal figures and $\tan\frac{\pi}{2} = 0$ by $1/0 = 0$. Therefore *two orthogonal figures can be considered to touch each other* [3], [4]. This implies that the line AB can be considered to touch the semicircles α, β and γ_1. Therefore it is appropriate to denote the line AB by γ_0 (see Figure 2).

Let γ be the semicircle of diameter AB constructed on the same side of AB as α. The area bounded by the semicircles α, β and γ is called an arbelos. Circles of radius ab/c are called Archimedean circles of the arbelos. Especially the Archimedean circle orthogonal to α and β, i.e., it touches AB at C, is called the Bankoff circle [1]. Therefore Theorem 1 shows that δ_1 is the Bankoff circle.

Acknowledgment. The author expresses his thanks to Professor Saburou Saitoh for offering the information γ_0 overlapping with the line AB stated in the last paragraph. Saburou Saitoh is the founder of division by zero $1/0 = 0$ and its generalization called division by zero calculus [4].

References

[1] L. Bankoff, Are the twin circles of Archimedes really twins?, Math. Mag., **72** (1974) 214–218.

[2] H. Okumura, A note on circles touching two circles in a Pappus chain, Sangaku J. Math., **6** (2022) 19–22.

[3] H. Okumura, Geometry and division by zero calculus, Int. J. Division by Zero Calculus, **1** (2021) 1–36.

[4] S. Saitoh, Introduction to the Division by Zero Calculus, 2021, Scientific Research Publ., Inc..