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# Some generalizations of five circle problems

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

**Abstract.** We generalize two problems in Wasan geometry involving three smaller congruent circles touching two larger congruent circles.

**Keywords.** congruent circles in line

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## 1. Introduction

If $\gamma_1$, $\gamma_2$, $\cdots$, $\gamma_n$ are congruent circles such that $\gamma_1$ and $\gamma_2$ touch, and $\gamma_i$ touches $\gamma_{i-1}$ at the farthest point on $\gamma_{i-1}$ from $\gamma_1$ for $i = 3, 4, \cdots, n$, then the circles are called *congruent circles in line* (see Figure 1). If $P$ (resp. $Q$) is the farthest point on $\gamma_1$ from $\gamma_2$ (resp. $\gamma_n$ from $\gamma_{n-1}$), then $P$ (resp. $Q$) is called the initial (resp. end) point. The line $PQ$ is called the axis.

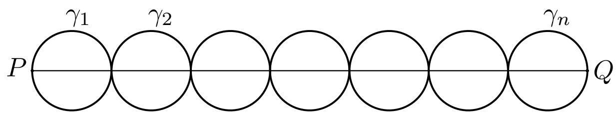


Figure 1.

In this article we give some generalizations of the two problems involving five circles proposed in [5] as Problems 3 and 4. The problems can be stated as follows (see Figures 2 and 3, respectively).

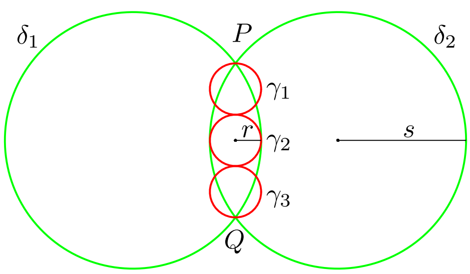


Figure 2: $s = 5r$.

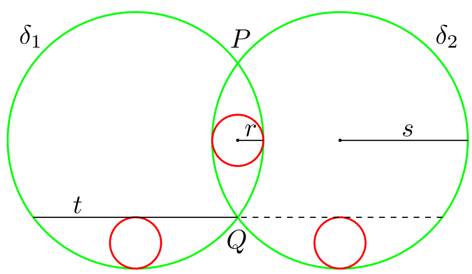


Figure 3: $s = 5r$.

**Problem 1.** For two circles $\delta_1$ and $\delta_2$ of radius $s$ meeting in points $P$ and $Q$, $\gamma_1$, $\gamma_2$, $\gamma_3$ are congruent circles in line of radius $r$ with initial point $P$ and end point $Q$, where the circle $\gamma_2$ touches $\delta_1$ and $\delta_2$ from inside of them. Show that $s = 5r$.

**Problem 2.** For two circles $\delta_1$ and $\delta_2$ of radius $s$ meeting in points $P$ and $Q$, the circle of radius $r$ and center at the midpoint of $PQ$ touches $\delta_1$ and $\delta_2$ from inside of them. If $t$ is the chord of $\delta_1$ overlapping with the perpendicular to $PQ$ at $Q$ and a circle of radius $r$ touches $t$ at the midpoint and the minor arc of $\delta_1$ cut by $t$, show that $s = 5r$.

Generalizations of similar problems can be found in [1, 2, 3, 4].

## 2. The case $s = |AB| + r$

In this paper, we consider a configuration consisting of a rectangle $ABCD$ with $|AB| \le |BC| = s$, and the circle $\delta$ of radius $s$ and center $A$ meeting the side $BC$ in a point $P$ (see Figure 4). We denote the configuration by $\mathcal{S}_P$, and will consider congruent circles in line of radius $r$ with the axis $BC$. In this section we consider the case in which only one circle of the congruent circles lies inside of $\delta$ and touches $\delta$ with the condition $s = |AB| + r$.

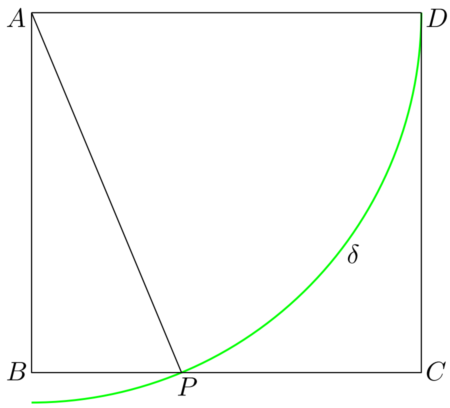


Figure 4: $\mathcal{S}_P$.

**Lemma 1.** *If $s = |AB|+r$ for $\mathcal{S}_P$, then $|BP| = (2z+1)r$ if and only if $|CP| = 2z^2r$ for a positive real number $z$. In this event $s = (z^2 + (z + 1)^2)r$ holds.*

*Proof.* Assume $s = |AB| + r$. If $|BP| = (2z + 1)r$, then $(s - r)^2 + ((2z + 1)r)^2 = s^2$ by the right triangle $ABP$. Solving the equation for $s$, we have $s = (z^2 + (z + 1)^2)r$. Hence $|CP| = s - |BP| = 2z^2r$. Conversely if $|CP| = 2z^2r$, then $(s - r)^2 + (s - 2z^2r)^2 = s^2$ by the same triangle. Solving the equation for $s$ we have $s = (z^2 + (z \pm 1)^2)r$. Since $(z^2 + (z - 1)^2)r < 2z^2r = |CP| < s$, we have $s = (z^2 + (z + 1)^2)r$ and $|BP| = s - |CP| = (2z + 1)r$. □

Problems 1 and 2 can be generalized as follows by Lemma 1 (see Figure 5).

**Theorem 1.** *Assume that circles $\delta_1$ and $\delta_2$ have radius $s$ and meet in points $P$ and $Q$ and $t$ is the chord of $\delta_1$ overlapping with the perpendicular to $PQ$ at $Q$. If the circle of radius $r$ and center at the midpoint of $PQ$ touches $\delta_1$ and $\delta_2$ from insides of them, then the following statements* (i) *and* (ii) *are equivalent.*
(i) *There are $2n + 1$ congruent circles in line of radius $r$ with initial point $P$ and end point $Q$.*

(ii) *There are $n^2$ congruent circles in line of radius $r$ such that its initial point coincides with the midpoint of $t$ and the end point coincides with the midpoint of the minor arc of $\delta_1$ cut by $t$.*
(iii) *If* (i) *or* (ii) *holds, then $s = (n^2 + (n+1)^2)r$ holds.*

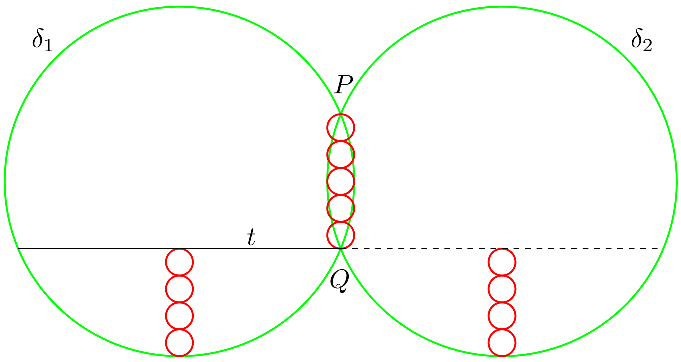


Figure 5: $n = 2$.

The theorem shows that the figures made by $\delta_1$ and $\delta_2$ in Problems 1 and 2 are congruent.

## 3. The case where two circles of radius $r$ lie inside of $\delta$

In this section we consider the case in which exactly two circles of the congruent circles in line lie inside of the circle $\delta$ and touch $\delta$. We get the next theorem (see Figure 6).

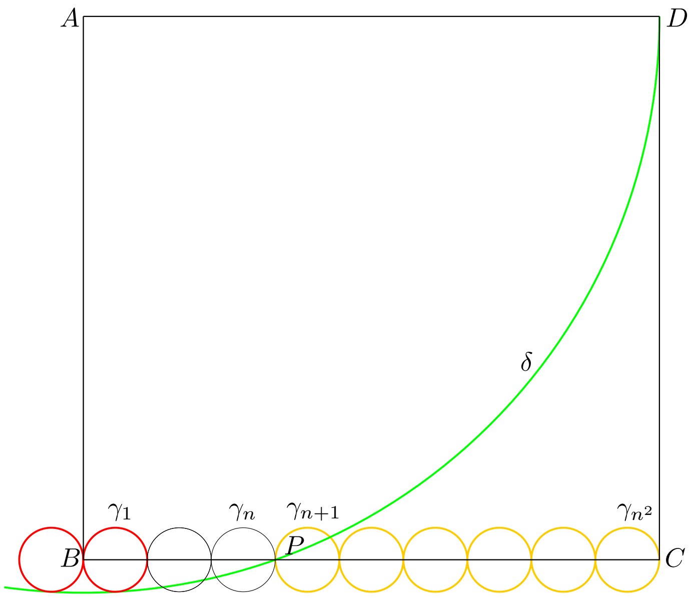


Figure 6: $n = 3$.

**Theorem 2.** *For $\mathcal{S}_P$, $\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots$ are congruent circles in line of radius $r$ with initial point $B$ such that the circle $\gamma_1$ touches $\delta$ and the center of $\gamma_1$ lies on the side $BC$. Then the following two statements* (i) *and* (ii) *are equivalent.*
(i) *$\gamma_1, \gamma_2, \cdots, \gamma_n$ are congruent circles in line with end point $P$.*

(ii) $\gamma_1, \gamma_2, \cdots, \gamma_{n^2}$ *are congruent circles in line with end point* $C$.
(iii) *If* (i) *or* (ii) *is true, the following relation holds.*

$$s = 2n^2 r. \qquad (1)$$

*Proof.* Assume (i). By the right triangles $ABP$ and the right triangle made by $A$, $B$ and the center of $\gamma_1$, we have $s^2 - (2nr)^2 = (s-r)^2 - r^2$. This gives (1), i.e., (ii) holds. Assume (ii). Then (1) holds. From the same right triangles, we have $(2n^2r)^2 - |BP|^2 = (2n^2r - r)^2 - r^2$. This gives $|BP| = 2nr$, i.e., (i) holds. □

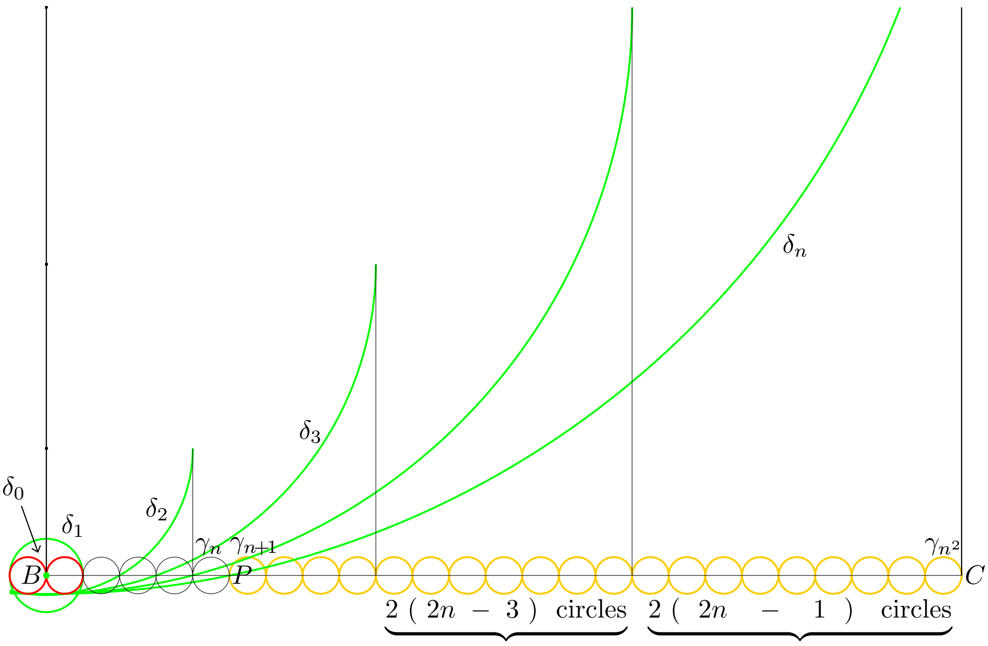


Figure 7: $n = 5$.

We explicitly denote the circle $\delta$ and the radius $s$ in Theorem 2 by $\delta_n$ and $s_n$. We can consider that $\delta_0$ is the point circle $B$ and $\delta_1$ is the circle of radius $2r$ and center $B$, and $s_n = s_{n-1} + 2(2n-1)r$ holds (see Figure 7).

## 4. THE CASE WHERE $n$ CIRCLES OF RADIUS $r$ LIE INSIDE OF $\delta$

In this section we consider the case where exactly $n$ circles of the congruent circles in line lie inside of $\delta$. We use the next theorem (see Figure 8).

**Theorem 3.** *For a point* $Q$ *on the segment* $BP$ *in* $\mathcal{S}_P$, *let* $\gamma_1, \gamma_2, \cdots, \gamma_n$ *be congruent circles in line with initial points* $P$ *and endpoint* $Q$. *Then the reflection of* $\gamma_1$ *in the point* $Q$ *touches* $\delta$ *internally if and only if* $|CP| = 2(2n-1)|BQ|$.

*Proof.* Let $r$ be the radius of $\gamma_1$. Then we obviously have

$$|BQ| + 2nr + |CP| = s.$$

Let $R$ be the center of the reflection of $\gamma_1$. By the right triangle $ABR$, we get

$$|AR|^2 = |AB|^2 + (|CP| + (4n-1)r - s)^2,$$

while from the right triangle $ABP$, we have

$$|AB|^2 = s^2 - (s - |CP|)^2.$$

Let

$$d = |AR|^2 - (s - r)^2.$$

Eliminating $s$, $|AB|$ and $|AR|$ from the four equations, we have

$$d = 2(|CP| - 2(2n - 1)|BQ|)r.$$

Therefore $|AR| = s - r$ and $|CP| = 2(2n - 1)|BQ|$ are equivalent. This proves the theorem. □

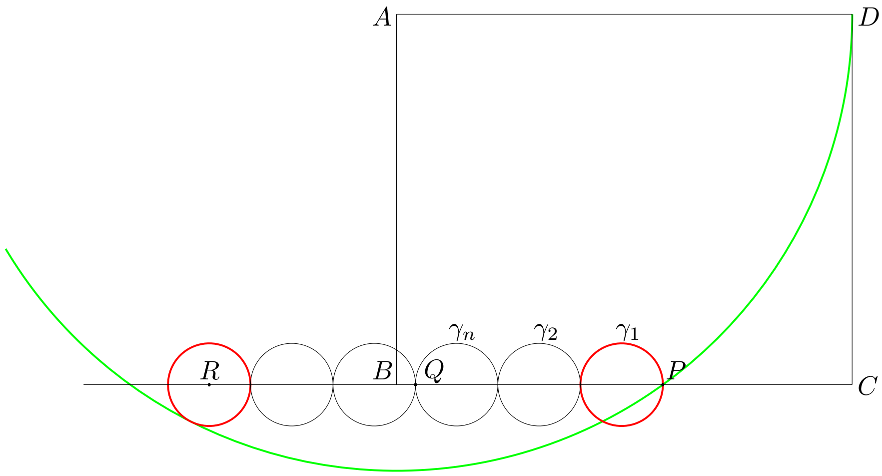


Figure 8: $n = 3$.

Considering the case $n = 1$ in Theorem 3, we get the next lemma.

**Lemma 2.** *Assume $Q$ is the point on the segment $BP$ such that $|PQ| = 2r$ for $\mathcal{S}_P$. If $\gamma_0$ is the circle of diameter $PQ$, and $\gamma'$ is the reflection of $\gamma_0$ in the point $Q$ and touches $\delta$, then*
(i) *$|BQ| = zr$ and $|CP| = 2zr$ are equivalent for a positive real number $z$.*
(ii) *If one of the relations in* (i) *holds, then $s = (3z + 2)r$.*

By Lemma 2, the next theorem holds (see Figures 9 for (i) of the theorem and see Figure 10 for (ii)). Problems 1 and 2 can be obtained if $n = 1$ in this theorem.

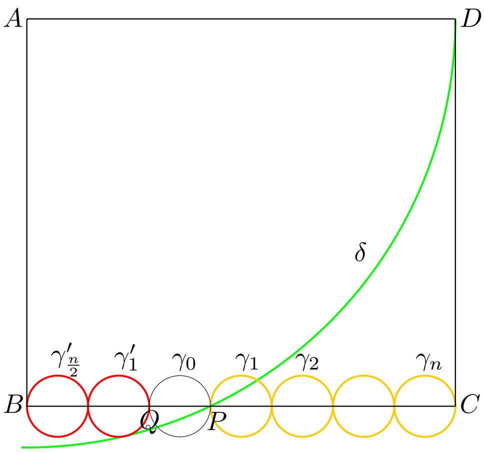


Figure 9: $n = 4$.

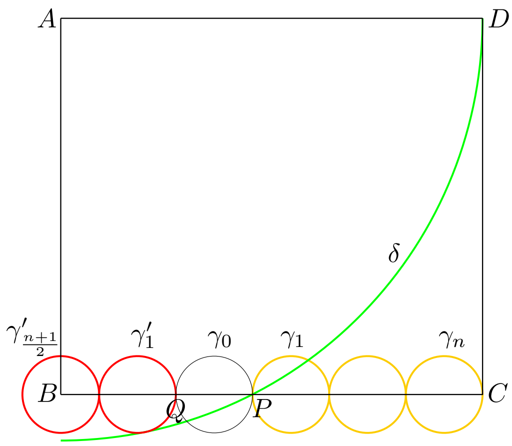


Figure 10: $n = 3$.

**Theorem 4.** *For a point $Q$ on the segment $BP$, $\gamma_0$ is the circle of diameter $PQ$, and $\gamma_0, \gamma_1, \gamma_2, \cdots$ are congruent circles in line of radius $r$, where $\gamma_1$ touches $\gamma_0$ at $P$. If $\gamma'_i$ is the reflection of $\gamma_i$ in the center of $\gamma_0$ and $\gamma'_1$ touches $\delta$, the following* *statements hold.*

(i) *If* $n$ *is even, then* $\gamma_1, \gamma_2, \cdots, \gamma_n$ *are congruent circles in line with end point* $C$ *if and only if* $\gamma'_1, \gamma'_2, \cdots, \gamma'_{\frac{n}{2}}$ *are congruent circles in line with end point* $B$*.*

(ii) *If* $n$ *is odd, then* $\gamma_1, \gamma_2, \cdots, \gamma_n$ *are congruent circles in line with end point* $C$ *if and only if the center of* $\gamma'_{\frac{n+1}{2}}$ *coincides with* $B$*.*

(iii) *If* (i) *or* (ii) *holds, then* $s = (3n + 2)r$ *holds.*

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