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Wasan Geometry and Division by Zero Calculus

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Abstract. In this paper, we will present some essential history and the logical points of the division by zero calculus. In order to see simply the results of the division by zero calculus, we will show the simple results in the typical and fundamental object triangles. Even the case of triangles, we will be able to derive new concepts and results from the division by zero property. In the second part of this paper, we will state several and typical applications to Wasan geometry. We will be able to see a great history and deep results of Wasan in the elementary geometry.

Keywords. division by zero, division by zero calculus, Wasan geometry

Mathematics Subject Classification (2010). 01A27, 03C99, 51M04

1. Global history on division by zero

The global history of the division by zero is given by [36] in details. There, it is stated that Aristoteles (BC384 - BC322) considered firstly the division by zero in the sense of physics, however, in a strict sense, Brahmagupta (598 - 668 ?) introduced zero and he already defined as $0/0 = 0$ in Brhmasphuasiddhnta (628). However, our world history stated that his definition $0/0 = 0$ is wrong over 1300 years, but, we showed that his definition is suitable. For the details, see the references and the site: http://okmr.yamatoblog.net/. In particular, we wrote the global book manuscript [40] with 211 pages.

We will recall the recent articles on the division by zero. J. A. Bergstra, Y. Hirshfeld and J. V. Tucker [6] and J. A. Bergstra [7] discussed the relationship between fields and the division by zero, and the importance of the division by zero for computer science. They, however, seem that the relationship of the division by zero and field structures are abstract.

Meanwhile, Carlström ([8]) introduced the wheel theory;

> wheels are a type of algebra where division is always defined. In particular, division by zero is meaningful. The real numbers can be extended to a wheel, as any commutative ring. The Riemann sphere can also be extended to a wheel by adjoining an element $\perp$, where $0/0 = \perp$. The Riemann sphere is an extension of the complex plane by an element ∞, where $z/0 = \infty$ for any complex $z \neq 0$. However, $0/0$ is still undefined on the Riemann sphere, but is defined in its extension to a wheel. The term wheel is introduced by the topological picture $\odot$ of the projective line together with an extra point $\perp = 0/0$.

Similarly, T.S. Reis and J.A.D.W. Anderson ([34, 35]) extend the system of the real numbers by defining division by zero with three infinities $+\infty, -\infty, \Phi$ (Transreal Calculus).

However, we can introduce simply a very natural field containing the division by zero that is a natural extension (modification) of our mathematics, as the Yamada field. For the above axiomatic great theories, it seems that some concrete examples derived from the theories are poor and they are abstract ones.

In connection with the deep problem with physics of the division by zero problem, see J. Czajko [10, 11, 12]. However, we will be able to find many logical confusions in the papers, as we refer to the essence later.

J. P. Barukčić and I. Barukčić ([5]) discussed the relation between the division $0/0$ and special relative theory of Einstein. However it seems that their result is curious with their logics. Their results contradict with ours.

L.C. Paulson stated that I would guess that Isabelle has used this **convention** $1/0 = 0$ since the 1980s and introduced his book [22] referred to this fact. However, in his group the importance of this fact seems to be entirely ignored at this moment as we see from the book.

For the more recent great works, see E. Jeřábek [14] and B. Santangelo [42]. They state in the abstracts of the papers as follows:

E. Jeřábek [14]:

> For any sufficiently strong theory of arithmetic, the set of Diophantine equations provably unsolvable in the theory is algorithmically undecidable, as a consequence of the MRDP theorem. In contrast, we show decidability of Diophantine equations provably unsolvable in Robinson's arithmetic Q. The argument hinges on an analysis of a particular class of equations, hitherto unexplored in Diophantine literature. We also axiomatize the universal fragment of Q in the process.

B. Santangelo [42]:

> The purpose of this paper is to emulate the process used in defining and learning about the algebraic structure known as a Field in order to create a new algebraic structure which contains numbers that can be used to define Division By Zero, just as i can be used to define $\sqrt{-1}$.

> This method of Division By Zero is different from other previous attempts in that each $\frac{\alpha}{0}$ has a different unique, numerical solution for every possible α, albeit these numerical solutions are not any numbers we have ever seen. To do this, the reader will be introduced to an algebraic structure called an S-Structure and will become familiar with the operations of addition, subtraction, multiplication and division in particular S-Structures. We will build from the ground up in a manner similar to building a Field from the ground up. We first start with general S-Structures and build upon that to S-Rings and eventually S-Fields, just as one begins learning about Fields by first understanding Groups, then moving up to Rings and ultimately to Fields. At each step along the way, we shall prove important properties of each S-Structure and of the operations in each of these S-Structures. By the end, the reader will become familiar with an S-Field, an S-Structure which is an extension of a Field in which we may uniquely define $\alpha/0$ for every non-zero α which belongs to the Field. In fact, each $\frac{\alpha}{0}$ has a different, unique solution for every possible α. Furthermore, this Division By Zero satisfies $\alpha/0 = q$ such that $0 \cdot q = \alpha$, making it a true Division Operation.

Meanwhile, we should refer to up-to-date information:

Riemann Hypothesis Addendum - Breakthrough Kurt Arbenz :

https://www.researchgate.net/publication/272022137 Riemann Hypothesis Addendum - Breakthrough.

Here, we recall Albert Einstein's words on mathematics:

> Blackholes are where God divided by zero. I don't believe in mathematics. George Gamow (1904-1968) Russian-born American nuclear physicist and cosmologist remarked that "it is well known to students of high school algebra" that division by zero is not valid; and Einstein admitted it as **the biggest blunder of his life** (Gamow, G., My World Line (Viking, New York). p 44, 1970).

We have still curious situations and opinions on the division by zero; in particular, the two great challengers Jakub Czajko [11] and Ilija Barukčić [5] on the division by zero in connection with physics stated recently that we do not have the definition of the division 0/0, however $0/0 = 1$. They seem to think that a truth is based on physical objects and is not on our mathematics. In such a case, we will not be able to continue discussions on the division by zero more, because for mathematicians, they will not be able to follow their logics more. However, then we would like to ask for the question that what are the values and contributions of your articles and discussions. We will expect some contributions, of course.

This question will reflect to mathematicians contrary. We stated for the estimation of mathematics in [33] as follows. Mathematics is the collection of relations and, good results are fundamental, beautiful, and give good impacts to human beings. With this estimation, we stated that the Euler formula

$$e^{\pi i} = -1$$

is the best result in mathematics in details in:

No.81, May 2012(pdf 432kb) www.jams.or.jp/kaiho/kaiho-81.pdf

In order to show the importance of our division by zero and division by zero calculus we are requested to show their importance. However, with the results stated in the references, we think the importance of our division by zero was already and definitely stated clearly.

It seems that the long and mysterious confusions for the division by zero were on the **definition**. – Indeed, when we consider the division by zero $a/0$ in the usual sense as the solution of the fundamental equation $0 \cdot z = a$, we have immediately the simple contradiction for $a \neq 0$, however, such cases $0/0$ and $1/0$ may happen, in particular, in mathematical formulas and physical formulas. The typical example is the case of $x = 0$ for the fundamental function $y = 1/x$.

– As we stated in the above, some researchers considered that for the mysterious objects $0/0$ and $1/0$, they considered them as ideal numbers as in the imaginary number i from its great success. However, such an idea will not be good as the number system, as we see simply from the concept of the Yamada field containing the division by zero.

Another important fact was **discontinuity** for the function $y = 1/x$ at the origin. Indeed, by the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear all as $a/0 = 0$ in the general fraction that is defined by the generalized solution of the equation $ax = b$. However, for the strong discontinuity of the function $y = 1/x$ at the origin, we were not able to accept the result $a/0 = 0$ for very long years.

As the number system containing the division by zero, the Yamada field structure is simple and complete. However for the applications of the division by zero to **functions**, we will need the concept of **division by zero calculus** for the sake of uniquely determinations of the results and for other reasons.

In this paper we will give our short history for the division by zero and we will refer to the concept of the division by zero calculus simply. In order to see simply the results of the division by zero, we will show the simple results in the typical and fundamental object triangles. Even the case of triangles, we will be able to derive new concepts and results from the division by zero property. The aim of this paper is to show that the division by zero is trivial and clear, however, elementary and fundamental for our mathematics. Furthermore, we will see a large new world and many fundamental open problems.

As the typical applications of the division by zero calculus, we will state the surprising results on the Wasan geometry. There, we will see a great history and deep results of Wasan in the elementary geometry.

2. Short history of division by zero

By a **natural extension** of the fractions b/a for any complex numbers a and b, we found the simple and beautiful result, for any complex number b

$$\frac{b}{0} = 0, \tag{1}$$

incidentally in [37] by the Tikhonov regularization for the Hadamard product inversions for matrices, and we discussed their properties and gave several physical interpretations on the general fractions in [16] for the case of real numbers. The result is a very special case for general fractional functions in [9].

Sin-Ei Takahasi ([16]) discovered a simple and decisive interpretation (1) by analyzing the extensions of fractions and by showing the complete characterization for the property (1):

Proposition 1. *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ satisfying*

$$F(b,a)F(c,d) = F(bc,ad) \quad for\ all \quad a,b,c,d \in \mathbf{C}$$

and

$$F(b,a) = \frac{b}{a}, \quad a,b \in \mathbf{C}, a \neq 0.$$

Then, we obtain $F(b,0) = 0$ for any $b \in \mathbf{C}$.

Note that the proposition is proved simply by 2 or 3 lines. In the long mysterious history of the division by zero, this proposition seems to be decisive.

Indeed, the Takahasi's assumption for the product property should be accepted for any generalization of fraction (division). Without the product property, we will not be able to consider any reasonable fraction (division).

Following the proposition, we should **define**

$$F(b,0) = \frac{b}{0} = 0,$$

and consider, for any complex number b, as (1); that is, for the mapping

$$W = \frac{1}{z},$$

the image of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([1]). As the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere ([20]). We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [49, 50, 51]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([20, 30]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidean geometry, analytic geometry, complex analysis and the theory of differential equations in an undergraduate level and furthermore to our basic ideas for the space and universe.

Meanwhile, the division by zero (1) was derived from several independent approaches as in:

1) by the generalization of the fractions by the Tikhonov regularization or by the Moore-Penrose generalized solution to the fundamental equation $az = b$ that leads to the definition of the fraction $z = b/a$,

2) by the intuitive meaning of the fractions (division) by H. Michiwaki,

3) by the unique extension of the fractions by S. Takahasi, as in the above,

4) by the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ into $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$,

and

5) by considering the values of functions with the mean values of functions.

Furthermore, in ([19]) we gave the results in order to show the reality of the division by zero in our world:

A) a field structure as the number system containing the division by zero — the **Yamada field Y**,

B) by the gradient of the y axis on the (x, y) plane — $\tan \frac{\pi}{2} = 0$,

C) by the reflection $W = 1/\overline{z}$ of $W = z$ with respect to the unit circle with center at the origin on the complex z plane — the reflection point of zero is zero, (The classical result is wrong, see [30]),

and

D) by considering rotation of a right circular cone having some very interesting phenomenon from some practical and physical problem.

In ([17]), we gave beautiful geometrical interpretations of determinants from the viewpoint of the division by zero. Furthermore, in ([20],[30]), we discussed many division by zero properties in the Euclidean plane - however, precisely, our new space is not the Euclidean space. More recently, we see the great impact to Euclidean geometry in connection with Wasan in ([26, 23, 28, 29]).

3. DIVISION BY ZERO CALCULUS

As the number system containing the division by zero, the Yamada field structure is complete. However for applications of the division by zero to **functions**, we will need the concept of division by zero calculus for the sake of uniquely determinations of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z - i}{z + i},$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z - (-i)},$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z - i) \cdot \frac{1}{z + i},$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot \left[\frac{1}{z+i}\right|_{z=-i} = (-2i) \cdot 0 = 0.$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples.

The short version of this section was given by [31] in the Proceedings of the International Conference:
`https://sites.google.com/site/sandrapinelas/icddea-2017`. In particular, the contents are mainly restricted to the differential equations for the sake of the conference topics.

Therefore, we will introduce the division by zero calculus: For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{2}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{3}$$

Note that here, there is no problem on any convergence of the expansion (2) at the point $z = a$, because all the terms $(z-a)^n$ are zero at $z = a$ for $n \neq 0$.

For the correspondence (3) for the function $f(z)$, we will call it **the division by zero calculus**. By considering the formal derivatives in (2), we can **define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

In order to avoid any logical confusion in the division by zero, we would like to refer to the logical essence:

For the elementary function $W = f(z) = 1/z$, we define $f(0) = 0$ and we will write it by $1/0 = 0$ following the form, apart from the intuitive sense of fraction. With only this new definition, we can develop our mathematics, through the division by zero calculus.

As a logical line for the division by zero, we can also consider as follows:

We define $1/0 = 0$ for the form; this precise meaning is that for the function $W = f(z) = 1/z$, we have $f(0) = 0$ following the form. Then, we can define the division by zero calculus for . In particular, from the function $f(x) \equiv 0$ we have $0/0 = 0$. In this sense, $1/0 = 0$ is more fundamental than $0/0 = 0$; that is, from $1/0 = 0$, $0/0 = 0$ is derived in this sense.

In order to avoid any logical confusion, we would like to state the essence, repeatedly.

Apart from the motivations above, we define the division by zero calculus by . With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as an axiom.

We defined the division by zero calculus for analytic functions, because we used the Laurent expansion. For the case of some smooth functions that are not analytical, the division by zero calculus is delicate. However, by applying the method of division by zero calculus by using the Taylor expansion for the Laurent expansion, we can consider the division by zero calculus. However, its logical situation is unclear and therefore we should check the results obtained. By checking the results obtained, we can enjoy the division by zero calculus for some general functions for creating new results. See [20].

We will give typical examples.

For the typical function $\sin x/x$, we have

$$\frac{\sin x}{x}(0) = \frac{\sin 0}{0} = \frac{0}{0} = 0,$$

however, by the division by zero calculus, we have, for the function $(\sin x)/x$

$$\frac{\sin x}{x}(0) = 1,$$

that is more reasonable in analysis.

However, for functions we see that the results by the division by zero calculus have not always practical senses and so, **for the results derived by division by zero calculus we should check the results, case by case.**

For example, for the line equation on the x, y plane

$$ax + by + c = 0,$$

we have, formally

$$x + \frac{by + c}{a} = 0,$$

and then, by the division by zero, we have, for $a = 0$, the reasonable result

$$x = 0.$$

Indeed, for the equation $y = mx$, from

$$\frac{y}{m} = x,$$

we have, by the division by zero, $x = 0$ for $m = 0$. This gives the case $m = \pm\infty$ of the gradient of the line as our common feelings. – This will mean that the equation $y = mx$ represents the general line through the origin in this sense. – This method was applied in many cases, for example see [26, 23].

However, from

$$\frac{ax + by}{c} + 1 = 0,$$

for $c = 0$, we have the contradiction, by the division by zero

$$1 = 0.$$

However, here, when we write as in

$$\frac{ax + by}{c} + \frac{c}{c} = 0$$

we see that this identity holds even for the case $c = 0$. Then, in this expression we can divide an equation by zero.

Meanwhile, note that for the function $f(z) = z + \frac{1}{z}$, $f(0) = 0$, however, for the function

$$f(z)^2 = z^2 + 2 + \frac{1}{z^2},$$

we have $f^2(0) = 2$. Of course,

$$f(0) \cdot f(0) = \{f(0)\}^2 = 0.$$

We consider the function

$$y = \frac{e^{cx}}{(c-a)(c-b)}.$$

If $c = a (\neq b)$, then, by the division by zero calculus, we have

$$y = \frac{xe^{ax}}{a-b}.$$

If $c = a = b$, then, by the division by zero calculus, we have

$$y = \frac{x^2 e^{ax}}{2}.$$

These functions have the practical meanings in the ordinary differential equations. See [31].

In the formula

$$\frac{x^{a+1}}{a+1}\left(\log x - \frac{1}{x+1}\right),$$

for $a = -1$, we have, by the division by zero calculus,

$$\frac{1}{2}\left(\log x\right).$$

The very important result

$$\tan \frac{\pi}{2} = 0$$

can be derived also analytically by the Laurent expansion and the expansion formula

$$\tan z = -\sum_{\nu=-\infty}^{+\infty}\left(\frac{1}{z-(2\nu-1)\pi/2} + \frac{1}{(2\nu-1)\pi/2}\right),$$

Furthermore, see [20] for many examples.

4. TRIANGLES AND DIVISION BY ZERO

In order to see how elementary of the division by zero, we will see the division by zero in triangles as a fundamental object.

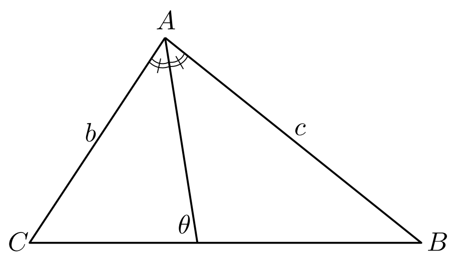

Figure 1.

We will consider a triangle ABC with side length a, b, c (see Figure 1). Let θ be the angle between the side BC and the angle bisector of A. Then, we have the identity

$$\tan\theta = \frac{c+b}{c-b}\tan\frac{A}{2}, \quad b < c.$$

For $c = b$, we have

$$\tan\theta = \frac{2b}{0}\tan\frac{A}{2}.$$

Of course, $\theta = \pi/2$; that is,

$$\tan\frac{\pi}{2} = 0.$$

Here, we used

$$\frac{2b}{0} = 0$$

and we do not consider that by the division by zero calculus

$$\frac{c+b}{c-b} = 1 + \frac{2b}{c-b}$$

and for $c = b$

$$\frac{c+b}{c-b} = 1.$$

Meanwhile, we have the formula

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{\tan B}{\tan C}.$$

If $a^2+b^2-c^2 = 0$, then $C = \pi/2$. Then,

$$0 = \frac{\tan B}{\tan\frac{\pi}{2}} = \frac{\tan B}{0}.$$

On the other hand, for the case $a^2-b^2+c^2 = 0$, then $B = \pi/2$, and we have

$$\frac{a^2+b^2-c^2}{0} = \frac{\tan\frac{\pi}{2}}{\tan C} = 0.$$

Let H be the perpendicular leg of A to the side BC, and let E and M be the mid points of AH and BC, respectively. Let θ be the angle of EMB ($b > c$). Then, we have

$$\frac{1}{\tan\theta} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

If $B = C$, then $\theta = \pi/2$ and $\tan(\pi/2) = 0$.

In the formula

$$\cos A = \frac{b^2+c^2-a^2}{2bc},$$

if b or c is zero, then, by the division by zero, we have $\cos A = 0$. Therefore, then we should understand as $A = \pi/2$.

This result is also valid in the Mollweide's equation

$$\sin\frac{B-C}{2} = \frac{(b-c)\cos\frac{A}{2}}{a},$$

for $a = 0$ as

$$0 = \frac{(b-c)\cos\frac{A}{2}}{0}.$$

5. Applications to Wasan geometry

Circles and lines are represented by the equation $a(x^2+y^2)+2gx+2fy+c=0$. If $a \neq 0$, then it represents a circle and its radius is given by

$$\sqrt{\frac{g^2+f^2-ac}{a^2}}.$$

Hence we can consider that the radius of a line equals 0 by the division by zero. The curvature of a line that is the reciprocal number of the radius is zero by the division by zero.

It is easy to consider that for a given circle γ with a tangent t the tangent of γ parallel to t is a circle of radius 0 touching γ and t. We see that the line perpendicular to γ and t, i.e., the perpendicular to t at the point of tangency of γ and t, is also a circle of radius 0 touching γ and t in a sense, because $\tan(\pi/2) = 0$. We consider the following theorem, which is a slight generalization of a theorem in Wasan geometry (see Figure 2) [2, 23].

Theorem 1. *If s and t are parallel lines touching a circle β of radius b, α is a circle of radius a touching β externally and s from the same side as β, γ is a circle of radius c touching α and β externally and t from the same side as β, then the following relation holds:*

$$c = \frac{b^2}{4a}. \tag{4}$$

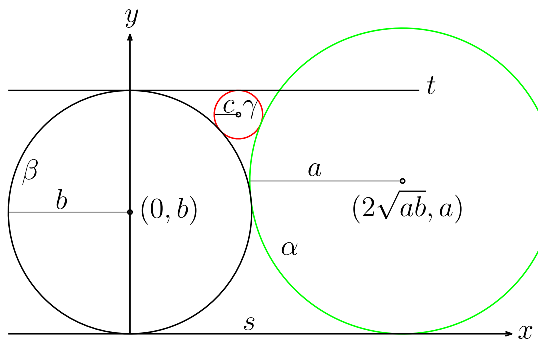

Figure 2.

We consider the case $a = 0$. Let O be the point of tangency of α and s. We use a rectangular coordinate system with origin O such that the center of β has coordinates $(0, b)$. We may assume that the circle α has an equation

$$(x - 2\sqrt{ab})^2 + (y-a)^2 - a^2 = 0.$$

The equation can be arranged as

$$\frac{x^2+y^2}{\sqrt{a}} - 4x\sqrt{b} - 2\sqrt{a}(y-2b) = 0,$$

$$\frac{x^2+y^2}{a} - 4x\sqrt{\frac{b}{a}} - 2(y-2b) = 0.$$

Therefore if $a = 0$, by the division by zero we have $x^2 + y^2 = 0$, $x = 0$ and $y - 2b = 0$, respectively, which express the origin, the y-axis and the line $y = 2b$ (the line t), respectively (see Figure 3).

The circle γ has an equation $(x - 2\sqrt{bc})^2 + (y - 2b + c)^2 = c^2$. Substituting $c = b^2/(4a)$ in this equation and rearranging, we get

$$a(x^2 + (y - 2b)^2) - 2bx\sqrt{ab} + \frac{b^2 y}{2} = 0,$$

$$\sqrt{a}(x^2 + (y - 2b)^2) - 2bx\sqrt{b} + \frac{b^2 y}{2\sqrt{a}} = 0,$$

$$(x^2 + (y - 2b)^2) - 2bx\sqrt{\frac{b}{a}} + \frac{b^2 y}{2a} = 0.$$

Hence if $a = 0$, we get $y = 0$, $x = 0$ and $x^2 + (y - 2b)^2 = 0$. The last three equations express the x-axis (the line s), the y-axis and the point $(0, 2b)$ (see Figure 4).

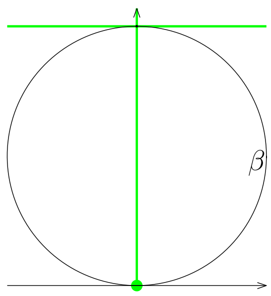

Figure 3: The circle α

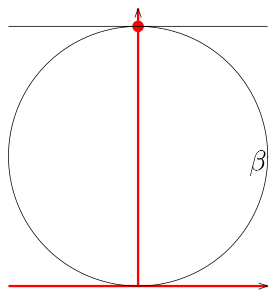

Figure 4: The circle γ

In any case $a = 0$ implies that γ is a line or a point, i.e., $c = 0$. Therefore (4) still holds in this case.

6. DESCARTES CIRCLE THEOREM

The following theorem was considered by Renè Descartes and is called the Descartes circle theorem with many references.

Theorem 2. *For mutually touching four circles γ_i $(i = 1, 2, 3, 4)$ of radii r_i, the following equation holds:*

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}. \tag{5}$$

This result and many variations were well-known in Wasan geometry [21, 43, 45, 47, 48]. In [26] we have shown that the theorem can also be considered in the degenerate cases in which some circles are points or lines [26], where one of the highlights is the case in which $\{\gamma_1, \gamma_2, \gamma_3\}$ consists of two proper circles and a point.

We now use a rectangular coordinate system such that γ_1 and γ_2 have equations $(x + r_1)^2 + y^2 = r_1^2$ and $(x - r_2)^2 + y^2 = r_2^2$, respectively. Let V_z be the point with coordinates $(0, 2\sqrt{r_1 r_2}/z)$ for a real number z. We use the next theorem.

Theorem 3 ([32])**.** *The circle passing through $V_{z\pm 1}$ for a real number $z \neq \pm 1$ and touching the circles C_1 and C_2 can be represented by the equation*

$$\left(x - \frac{r_1 - r_2}{z^2 - 1}\right)^2 + \left(y - \frac{2z\sqrt{r_1 r_2}}{z^2 - 1}\right)^2 = \left(\frac{r_1 + r_2}{z^2 - 1}\right)^2. \tag{6}$$

Assume that the circle C_3 is represented by (6), $z = 1/w$ and C_4 has an equation

$$(x - x_4)^2 + (y - y_4)^2 = r_4^2. \tag{7}$$

Since C_4 touches C_1, C_2 and C_3, we get

$$x_4 = r_1 r_2 (r_1 - r_2) w^2 / D,$$

$$y_4 = 2 r_1 r_2 (\sqrt{r_1 r_2} + (r_1 + r_2) w) w / D,$$

$$r_4 = r_1 r_2 (r_1 + r_2) w^2 / D,$$

where

$$D = r_1 r_2 + 2\sqrt{r_1 r_2}(r_1 + r_2) w + (r_1^2 + r_1 r_2 + r_2^2) w^2.$$

Notice that there are four sets of the solutions of x_4, y_4, r_4, but we consider one set, because the other cases can be considered similarly.

Substituting these three in (7), we obtain

$$f = f_0 + f_1 w + f_2 w^2 = 0,$$

where $f_0 = r_1 r_2 (x^2 + y^2)$, $f_1 = 2\sqrt{r_1 r_2}((r_1 + r_2)(x^2 + y^2) - 2 r_1 r_2 y)$, and $f_2 = (r_2^2 + r_1 r_2 + r2^2)(x^2 + y^2) + 2 r_1 r_2 (r_2 - r_1) x - 4 r_1 r_2 (r_1 + r_2) y + 4 r_1^2 r2^2$. By the division by zero calculus at $w = 0$ for $f = 0$, $f/w = 0$ and $f/w^2 = 0$, we obtain

$$x^2 + y^2 = 0, \tag{8}$$

$$(r_1 + r_2)(x^2 + y^2) - 2 r_1 r_2 y = 0, \tag{9}$$

$$(r_1^2 + r_1 r_2 + r_2^2)(x^2 + y^2) + 2 r_1 r_2 (r_1 - r_2) x - 4 r_1 r_2 (r_1 + r_2) y + 4 r_1^2 r_2^2 = 0. \tag{10}$$

The equation (8) expresses the origin (the point of tangency of C_1 and C_2) (see Figure 5). The equation (9) expresses the circle of radius $r_1 r_2/(r_1 + r_2)$ orthogonal to C_1 and C_2, which is called the Bankoff circle of the figure made by C_1, C_2 and the smallest circle touching them internally. Each of the figures surrounded by C_1, C_2 and this circle is called an arbelos, and the equation (10) expresses the incircle of the arbelos. Notice that the Bankoff circle is also orthogonal to this circle.

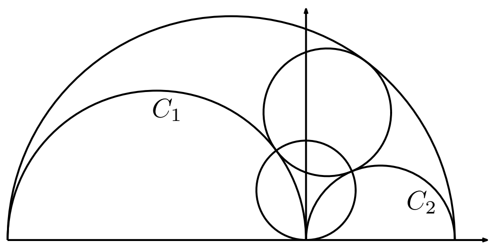

Figure 5.

If C_4 is expressed by (8), then (5) does not hold. However if C_4 is expressed by (9), then (5) is true for $r_3 = 0$. For if $\xi = \sqrt{r_3}$, then the right side of (5) equals

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\xi^2} \pm \frac{2}{\xi}\sqrt{\frac{\xi^2}{r_1 r_2}\frac{1}{r_1} + \frac{1}{r_2}} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2}.$$

If C_4 is expressed by (10), then (5) is also true if C_3 forms an arbelos with C_1 and C_2, which is obtained from (6) in the case $z = 0$.

7. A LEMMA IN SAMPŌ JOJUTSU

We consider the following lemma of Wasan geometry [46] (see Figure 6).

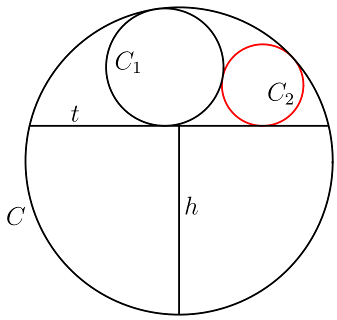

Figure 6.

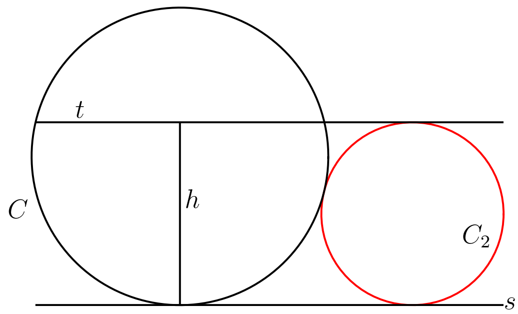

Figure 7.

Lemma 7.1. *Assume that the circle C of radius r is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t. If two externally touching circles C_1 and C_2 of radii r_1 and r_2 also touch the chord t and the other arc of the circle C internally, then h, r, r_1 and r_2 are related by*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}. \tag{11}$$

Let s be the tangent of C parallel to t touching C at the point whose distance from t is equal to h. In [29], we have considered the circle C_2 in the case C_1 being the point of intersection of C and t in this lemma by division by zero calculus. The most intersecting case obtained in this case is that C_2 coincides with one of the circles touching C externally and s and extended t (see Figure 7). For if $r_2 = -h/2$, then the left side of (11) equals

$$\frac{1}{0} + \frac{1}{r_2} + \frac{2}{h} = 0.$$

Therefore Lemma 7.1 still holds, since the right side of (11) also equals

$$2\sqrt{\frac{2r}{0 \cdot r_2 h}} = 0.$$

Notice that the minus sign of r_2 can be justifiable because C_2 touches C externally, while C_1 and C_2 touch C internally in the ordinary case.

For more examples of applications of the definition of division by zero to Wasan geometry and related topics see [15, 24, 25, 26, 27, 28].

8. Conclusion and open problems

The essential problems with the mysterious history of the division by zero were on the **definition** of the division by zero and the strong **discontinuity** of the fundamental function $y = 1/x$ at the origin. This discontinuity was not accepted for long years. One more problem for the division by zero is on the concept of the **division by zero calculus**; that is, the fractions and functions cases are different, as we showed clearly.

We have considered our mathematics around an isolated singular point for analytic functions, however, we did not consider mathematics **at the singular point itself**. At the isolated singular point, we considered our mathematics with the limiting concept, however, the limiting values to the singular point and **the values at the singular point** in the sense of division by zero calculus are different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. We therefore have a general open problem discussing our mathematics on a domain containing the singular points.

We stated, on the division by zero, the importance of the definition of the division by zero $z/0$. However, we note that in our definition it is given as a **generalization** or **extension** of the usual fraction. Therefore, we will not be able to give its precise meanings at all. For this sense, we do not know the direct meaning of the division by zero. It looks **like a black hole**. In order to know its meaning, we have to examine many properties of the division by zero calculus by applications.

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