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A note on an isosceles triangle containing a square and three congruent circles

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Abstract. A problem involving an isosceles triangle containing a square and three congruent circles is generalized.

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1. Introduction

In this note we generalize the following problem, which can be found in [1, 2, 3, 4, 5], where the sangaku with this problem in [4] is undated (see Figure 1).

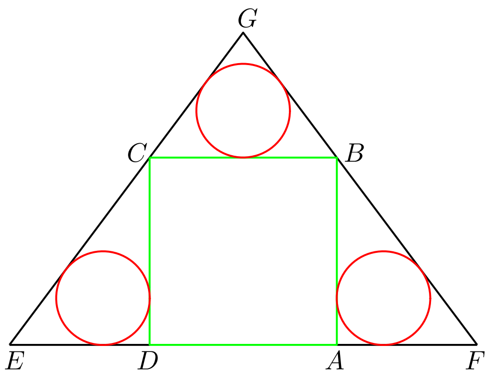

Figure 1.

Problem 1. EFG is an isosceles triangle with base EF. $ABCD$ is a square such that B and C lie on the sides FG and GE, respectively, and D and A lie on the side EF. If the incircles of the triangles ABF and BCG are congruent and have radius r, show that $4r = |AB|$.

We show that the isosceles triangle EFG is formed by a 3-4-5 triangle with its reflected image in the side of length 4, i.e., the ratio of the sides of EFG equals $5 : 5 : 6$ in a more general situation.

2. GENERALIZATION

Let EFG be a triangle. Let γ_1, γ_2, $\cdots$, γ_n be circles of radius r such that they touch the side EF from the inside of EFG, γ_1 and γ_2 touch, γ_i $(i = 3, 4, \cdots, n)$ touches γ_{i-1} from the side opposite to γ_1, γ_1 touches the side GE, γ_n touches the side FG (see Figure 2). In this case we say that EF *has* n *circles of radius* r *with respect to* G. This is equivalent to the following equation being true:

$$|EF| = r \cot \frac{\angle E}{2} + r \cot \frac{\angle F}{2} + 2(n-1)r.$$

Problem 1 is generalized as follows (see Figure 3).

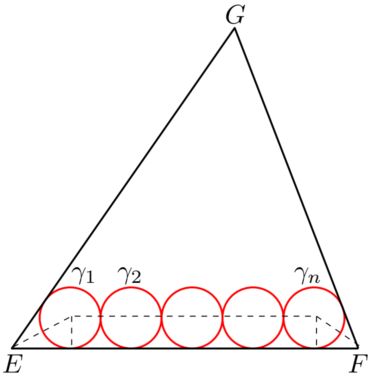

Figure 2: $n = 5$

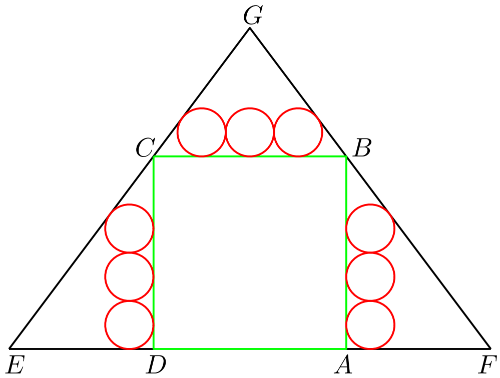

Figure 3: $n = 3$

Theorem 2.1. *EFG is an isosceles triangle with base EF. $ABCD$ is a square such that B and C lie on the sides FG and GE, respectively, D and A lie on the side EF. If AB has n circles of radius r with respect to F and BC has n circles of radius r with respect to G, then the following statements hold.*
(i) $|FG| : |EF| = 5 : 6$.
(ii) $2(n+1)r = |AB|$.
(iii) *If n is odd and expressed as $n = 2k-1$ for a natural number k, EF has $5k-1$ circles of radius r with respect to G.*

Proof. Let $2\theta = \angle ABF$. Then we have

$$|AB| = r \cot \theta + (2n-1)r. \tag{1}$$

While $\angle CBG + 2\theta = 90°$ implies $|BC| = 2r\cot(45° - \theta) + 2(n-1)r$. Therefore we get $\cot\theta = 3$ by $|AB| = |BC|$. Hence $\tan 2\theta = 3/4$, i.e., ABF is a 3-4-5 triangle. This proves (i). The part (ii) follows from (1). We assume $n = 2k-1$. Let $s = |AB|$. Then $s = 4kr$ by (ii). The distance from G to BC equals $s \cdot (4/6) = 2s/3$. Therefore $|BC| : |EF| = 2s/3 : (s + 2s/3) = 2 : 5$, i.e., $|EF| = 5s/2 = 10kr$. Hence $|EF| = 2r\cot(\angle E/2) + 2(5k-1-1)r$, since $\cot(\angle E/2) = 2$. This proves (iii). □

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Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.